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A General Theory of the Unit Hydrograph

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Abstract—By the single assumption that the reservoir action in a catchment can be separated from translation, the general equation of the unit hydrograph is shown to be

$$u(0, t) = \frac{V_0}{A} \int_0^{A(t)} \frac{\delta(t - \tau)}{\Pi(1 + K_i D)} \cdot i \cdot dA$$

This is simplified by two further simple assumptions to give

$$u(0, t) = \frac{V_0}{T} \int_0^{t \leq T} P(m, n - 1) \cdot \omega(\tau') \cdot dm$$

which can be conveniently calculated.

LIST OF SPECIAL SYMBOLS

A—area of catchment
  cross section of channel (Section 3)
$a(x)$—translation time
$I(t)$—inflow rate
K—delay time of reservoir storage
M— -*th* moment of the unit hydrograph
N—total number of reservoirs in the catchment
m—dimensionless time variable
$n(\tau)$—distribution of reservoirs in the catchment
$P(m, n)$—Poisson probability function
$Q(t)$—outflow rate
$S(t)$—storage volume
T—maximum translation time in catchment
$U(0, t)$—instantaneous unit hydrograph
$U(D, t)$—finite period unit hydrograph
V_0—volume of rainfall excess (outflow)
$\delta(0)$—Dirac-delta function
Π—product of similar terms to be taken
τ—total translation delay time between point and outlet
$\omega(\tau)$—ordinate of time-area-concentration curve
i—$\dfrac{\text{local rainfall intensity}}{\text{average rainfall intensity}}$

INTRODUCTION

Since its first proposal over 25 years ago [*Sherman*, 1932], the unit hydrograph approach to stream flow prediction has developed into one of the most powerful tools of applied hydrology. It has, however, retained an empirical character, and no general theoretical basis for the method has been evolved. The absence of a general theory of the unit hydrograph has limited the scope of the method and made it dependent on personal judgment in its practical application.

The present paper proposes a general theory which should help to remove many of the subjective elements from unit hydrograph analysis, and also to release the problem of synthesis from its present dependence on empirical relationships derived from localized data. To be worthwhile, a theory should enable us to predict the effect on the unit hydrograph of such variables as the shape of the catchment, the speed of travel of flood waves in the catchment, the amount and distribution of both channel storage and overbank storage, unequal rainfall distribution over the catchment, the size and slope of the catchment, and so on. It would be gratifying, also, if such a theory could (a) provide a simple objective method of deriving a unit hydrograph from complex storms; and (b) indicate a small number of physically significant parameters which could be correlated with catchment characteristics to form the basis of a universally applicable system of unit hydrograph synthesis.

Zoch [1934, 1936, 1937] made a notable attempt to formulate a general physical theory of stream flow based on the assumption that at any time the rate of discharge is proportional to the amount of rainfall remaining with the soil at that time. Using this assumption, he analysed the runoff due to rainfall of finite duration and uniform rate, obtaining equations for the four separate segments of the hydrograph. Zoch solved these equations for two simple cases, where the time-area-concentration curve is rectangular or triangular. He indicated as the main difficulty in the solution of other cases the integration of $\omega(x)e^{Kx}$ where $\omega(x)$ is the ordinate of the time-area-concentration curve and K is a constant; he suggested the use of series approximation or numerical integration in such cases.

Clark [1945] suggested that the unit hydrograph for instantaneous rainfall could be derived by routing the time-area-concentration curve through a single element of linear reservoir storage. Physically this is equivalent to Zoch's formulation, the equations being simplified by reducing rainfall duration to zero, and the reservoir routing being a form of numerical integration of $\omega(x)e^{Kx}$. The hydrology group of the Irish Office of Public Works found that in many practical cases the smoothing involved in routing was sufficient to permit the replacement of the tediously derived time-area-concentration curve by an isosceles triangle [*O'Kelly*, 1955]. The present author in a thesis study discussed the physical basis of the Clark approach and the O'Kelly simplification [*Dooge*, 1956].

A further hypothesis was put forward by *Nash* [1957], more as the basis of a convenient two-parameter synthetic method than as a general theory. He suggested that the instantaneous unit hydrograph could be derived by routing the instantaneous rainfall through a series of successive linear reservoirs of equal delay time.

In the present paper, a general equation for the unit hydrograph is derived from the single physical assumption that the reservoir action which takes place in the catchment can be separated from the translatory action and lumped in a number of reservoirs unrestricted in number, size, or distribution. This general equation can be converted from a surface integral to a simple integral by assuming that above each confluence in the catchment the reservoirs are equally distributed for equal lengths of tributary. The complexity of the computation required is greatly reduced if the idealised reservoirs in the catchment are assumed equal, since in this case the ordinates of the unit hydrograph can be obtained by integrating the product of an ordinate of the time-area-concentration curve and an ordinate of the Poisson probability function (which has been tabulated). Even with this assumption of equal reservoirs, the result is extremely general and flexible, containing each of the previous theoretical approaches as a very special case.

Unit hydrograph principles

All unit hydrograph procedures (analytical or synthetic) are based on two fundamental principles:

1. Invariance—the hydrograph of surface runoff from a catchment due to a given pattern of rainfall excess (that is, rainfall minus infiltration and similar losses) is invariable.

2. Superposition—the hydrograph resulting from a given pattern of rainfall excess can be built up by superimposing the unit hydrographs due to the separate amounts of rainfall excess occurring in each unit period. This includes the principle of proportionality, by which the ordinates of the hydrograph are proportional to the volume of rainfall excess.

Further assumptions are often made for practical convenience, but they are not essential. Thus the starting point for Sherman's original work on the subject was the assumption that all floods due to rainstorms of a given duration ran off in the same amount of time. This use of a finite time base for the unit hydrograph is not essential and is only physically reasonable for cases where the storage is distributed evenly throughout the catchment.

For the purpose of the present study the following definitions are used. The instantaneous unit hydrograph for a catchment, $u(0, t)$ is the hydrograph of surface runoff at the outlet from the catchment due to a finite volume of rainfall excess, V_0, falling in an infinitesimally short time. Such a rainfall excess is written in terms of the δ-function as $V_0 \cdot \delta(0)$. A D-period unit hydrograph, $u(D, t)$, is the hydrograph of surface runoff at the outlet due to a rainfall excess of volume, V_0, distributed uniformly throughout a period, D. In both cases the areal distribution of rainfall excess throughout the catchment is assumed to follow a constant pattern from storm to storm, but this pattern is not necessarily one of uniform distribution. The conversion from an instantaneous unit hydrograph to one of finite duration can be easily made, since by superposition

$$u(D, t) = \frac{1}{D} \int_{t-D}^{t} u(0, t) \cdot dt$$

Unless the contrary is stated, all unit hydrographs discussed in the present paper are instantaneous unit hydrographs whose ordinates have the dimension of discharge rate per unit area.

The presence of the principle of superposition implies that any theory of the unit hydrograph must be a linear one. The processes involved in the conversion of rainfall excess to surface runoff at the outlet must all be linear if the unit hydrograph theory is to hold exactly, since the presence of a single non-linear element would be sufficient to destroy the principle of superposition. It is clear that any general theory must be a linear one, and that cases of catchments containing non-linear elements must be dealt with by some process of linearization.

The key problem, therefore, is to determine the equation of the instantaneous unit hydrograph for a catchment containing only linear elements. The corresponding problem in applied mathematics is the determination for a linear system of the Green's function, that is, the response of the system to the δ-function or unit impulse. The first step towards the solution of the hydrological problem is to examine the response of simple linear elements to an instantaneous inflow.

Once the instantaneous unit hydrograph is known, the runoff due to any given rainfall pattern can be found by convolution,

$$q = i(t) * u(0, t)$$

$$= \int_0^t u(0, t - \tau) \cdot i(\tau) \cdot d\tau$$

In practice this final calculation is most conveniently made by using a distribution graph, or by using a movable strip technique.

Linear channels and reservoirs

The process of converting rainfall excess into surface runoff is a mixture of translation and reservoir action. The first step to be taken is to examine the cases of pure reservoir action and pure translation in a linear catchment.

A linear reservoir is one in which the storage is directly proportional to the outflow:

$$S(t) = KQ(t)$$

the constant K having the dimension of time and being equal to the average delay time imposed on an inflow by the reservoir. If the above relationship is combined with the storage equation

$$I(t) - Q(t) = \frac{d}{dt} S(t)$$

we get

$$I(t) - Q(t) = K \frac{d}{dt} Q(t)$$

which can be written in operational form as

$$(1 + KD)Q(t) = I(t)$$

where D is the differential operator. This equation has the solution

$$Q(t)e^{t/K} = \frac{1}{K} \int I(t)e^{t/K}\, dt + \text{constant}.$$

For any given inflow this equation can be solved either analytically or numerically. For an instantaneous inflow the outflow from a linear reservoir is given by

$$Q(t) = e^{-t/K}\left[\int \frac{V_0}{K} e^{t/K}\, \delta(0) \cdot dt + \text{const.}\right]$$

$$= \frac{V_0}{K} e^{-t/K}, \qquad t > 0$$

so that the response of a linear reservoir to the δ-function is a sudden jump at the instant of inflow to a finite outflow followed by an exponential decline approaching infinity.

In order to discuss the problem of translation, it is convenient to introduce the concept of a linear channel. Such a channel is analogous to the idea of a linear reservoir whose storage outflow curve is a straight line. A channel unaffected by backwater has a definite rating curve at every point of the channel. A linear channel is defined as a reach in which the rating curve at every point is a linear relationship between discharge and area. This implies that at any point the velocity is constant for all discharges, but may vary from point to point along the reach. If the linear rating is written as

$$A = a'(x) \cdot Q$$

this equation can be applied to any point along the linear channel. This relationship can be combined with the continuity equation

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

giving the equation

$$\frac{\partial Q}{\partial x} + a'(x)\cdot\frac{\partial Q}{\partial t} = 0$$

which has the solution

$$Q[t - a(x)] = \text{constant}.$$

This solution corresponds to the case of pure translation. It indicates that a linear channel will translate any inflow hydrograph without change of shape. This is in contrast with the non-linear case in which only one inflow hydrograph is capable of uniform translation, that is, the special case of uniformly progressive flow. For the case of instantaneous inflow at the upstream end of a linear channel, the flow at any other point of the channel is given by

$$Q = V_0\cdot\delta(a)$$

Combinations of Channels and Reservoirs

If a linear reservoir and a linear channel are placed in series, the order in which they occur is immaterial, since the translation due to the linear channel merely involves the shifting of a time scale. The outflow from two such elements in series due to an instantaneous inflow $V_0\cdot\delta(0)$ is given by

$$t \geq a \qquad Q = \frac{V_0}{K}e^{-(t-a)/K}$$

where K is the delay time due to the linear reservoir and a is the translation time due to the linear channel.

If a number of linear channels are placed in series, the effect is merely one of translation by an amount equal to the sum of the translation times of the individual linear channels.

If a number of unequal linear reservoirs occur in series, the outflow from one is the inflow to the next, so that the equation is

$$(1 + K_1D)(1 + K_2D) \cdots (1 + K_nD)Q_n(t) = I(t)$$

or

$$Q_n(t) = \frac{1}{(1+K_1D)(1+K_2D)\cdots(1+K_nD)}\,I(t)$$

which can be written as

$$Q_n(t) = \frac{1}{\Pi_1{}^n(1 + K_iD)}\cdot I(t)$$

For the instantaneous inflow this reduces to

$$t > 0 \qquad Q_n(t) = \frac{1}{\Pi(1 + K_iD)}\cdot\{0\}$$

which is easily solved, since the inverse operator is already divided into factors.

If all the linear reservoirs are unequal, the outflow must be of the form

$$Q_n(t) = C_1e^{-t/K_1} + C_2e^{-t/K_2} + \cdots + C_ne^{-t/K_n}$$

Since the flow from any one of the reservoirs decreases to zero after an infinite time, we can write for each reservoir

$$\int_0^\infty Q_i\cdot dt = V_0 \qquad i = 1, 2 \cdots n$$

These n equations form the boundary conditions which enable us to evaluate the n unknown coefficients $C_1 \cdots C_n$. The insertion of these boundary conditions gives the set of equations

$$\int_0^\infty Q_n\,dt = V_0$$

i.e.

$$\int_0^\infty (C_1e^{-t/K_1} + C_2e^{-t/K_2} + \cdots C_ne^{-t/K_n})\cdot dt = V_0$$

$$\int_0^\infty Q_{n-1}\cdot dt = V_0$$

$$\int_0^\infty (1 + K_nD)Q_n\cdot dt = V_0$$

$$\int_0^\infty Q_n\cdot dt + \int_0^\infty K_n\frac{d}{dt}Q_n\cdot dt = V_0$$

$$V_0 + K_n[Q_n]_0^\infty = V_0$$

i.e.

$$[C_1e^{-t/K_1} + C_2e^{-t/K_2} + \cdots C_ne^{-t/K_n}]_0^\infty = 0$$

etc. etc.

which reduce to the algebraic equations

$$C_1K_1 + C_2K_2 + \cdots + C_nK_n = V_0$$

$$C_1 + C_2 + \cdots + C_n = 0$$

$$\frac{C_1}{K_1} + \frac{C_2}{K_2} + \cdots + \frac{C_n}{K_n} = 0$$

$$\cdots\cdots\cdots\cdots\cdots = 0$$

$$\frac{C_1}{K_1^{n-2}} + \frac{C_2}{K_2^{n-2}} + \cdots + \frac{C_n}{K_n^{n-2}} = 0$$

On solving these, the general equation can be written as

$$\frac{Q_n(t)}{V_0} = \frac{K_1^{n-2}e^{-t/K_1}}{\Pi(K_1 - K_i)} + \frac{K_2^{n-2}e^{-t/K_2}}{\Pi(K_2 - K_i)} + \cdots + \frac{K_n^{n-2} \cdot e^{-t/K_n}}{\Pi(K_n - K_i)}$$

so that the outflow from a chain of unequal linear reservoirs can be expressed as a sum of n terms, each of which is a simple exponential decay curve. This does not furnish us with an easily manipulated tool for analysis and subsequent synthesis, since the separation of exponentials by numerical analysis is an unsatisfactory process [*Lanczos*, 1957]. If some of the n linear reservoirs are equal, repeated roots are obtained. This complicates some of the individual terms without reducing the number of terms in the series.

In the special case where all of the n linear reservoirs are equal, a remarkable simplification is obtained. In this case we have

$$Q_n(t) = (1 + KD)^{-n}\{0\}$$
$$= (C_1 + C_2t + C_3t^2 + \cdots + C_nt^{n-1})e^{-t/K}$$

Insertion of the boundary conditions eventually results in the following

$$C_1 = C_2 = C_3 = \cdots = C_{n-1} = 0$$

$$C_n = \frac{V_0}{K^n(n-1)!} \quad .$$

so that the outflow is given by

$$Q_n(t) = \frac{V_0}{K}\frac{(t/K)^{n-1}e^{-t/K}}{(n-1)!}$$

which shows that the outflow is represented by a single term. The fact that the term is complicated is of no practical significance, since it is identical with the Poisson probability function, values of which have been tabulated by *Pearson* [1930], and *Molina* [1942]. This fact is taken advantage of later in the paper to facilitate a numerical solution of the general equation of the unit hydrograph. The outflow from n equal reservoirs can, therefore, be written conveniently as

$$Q_n(t) = \frac{V_0}{K} P\left(\frac{t}{K}, n - 1\right)$$

where P is the Poisson probability function.

One final case is of special interest. If a total delay time of K is assumed, the effect on the outflow hydrograph of varying the value of n gives some interesting results. The maximum outflow occurs at a time $(n - 1)K$ and has the value

$$Q_{max} = \frac{V_0}{K} \cdot P(n - 1, n - 1)$$

As n increases, the peak increases and other values decrease until for n equal to infinity the outflow is $V_0 \cdot \delta(\overline{n - 1} \cdot K)$. This result indicates that if a finite total delay time is divided up among a large number of equal linear reservoirs, the effect on an instantaneous inflow is equivalent to translation by an amount equal to the total delay time.

Thus we can look on (1) reservoir action as being related to concentrated storage and (2) translatory action as related to completely distributed storage.

A GENERAL EQUATION FOR THE UNIT HYDROGRAPH

The linear components already discussed can be combined to represent an ideal linear catchment and an equation written for the unit hydrograph for such a catchment. This is done by assuming that any linear catchment can be represented by an ideal linear catchment in which the storage (both channel and overbank) is either concentrated or completely distributed; that is, the ideal catchment is drained by a network of channels composed of a complex system of linear channels and linear reservoirs placed in series. This lumping of the reservoir action is the only assumption made in deriving the general equation (apart from the necessary assumption of linearity without which no true unit hydrograph would exist).

The first step is to consider the outflow from an element of area in the idealized catchment. Since the whole system is linear, the contribution of any portion can be considered in isolation. To reach the outlet the rainfall excess falling on any element of area must pass through a linear reservoir representing either overland flow or interflow, and then pass through the chain of linear channels and linear reservoirs representing the drainage system between the point under consideration and the outlet. Since the order of operations is immaterial, the flow may be considered as first passing through a succession of linear channels and then through the series of linear reservoirs appropriate to the portion of the catchment under examination. For a rainfall excess $I(t)$ on an element of catchment ΔA the resulting outflow is given by

$$\Delta Q(t) = \frac{1}{\Pi_1{}^n(1 + K_i D)} \cdot \{I(t - \tau)\} \cdot \Delta A$$

where

τ = total translation time between the element and the outlet

and

$K_1 \cdots K_n$ = storage delay times of reservoirs between the element and the outlet.

From this it follows that the contribution of the element of area ΔA to the instantaneous unit hydrograph is given by

$$t < \tau \qquad \Delta u(0, t) = 0$$

$$t \geq \tau \qquad \Delta u(0, t) = \frac{1}{\Pi(1+K_i D)} \; V_0 \delta(t-\tau) \; \frac{\Delta A}{A}$$

If the value of τ and the values of the n storage delay times $K_1 \cdots K_n$ are known, this equation can be calculated as indicated in the preceding section.

The total runoff from the catchment after a time t will clearly be the sum of the contributions of the individual elements of area. It is obvious that those areas, where the total translation time τ is greater than the elapsed time t since the start of rainfall excess, will contribute nothing to the runoff at the outlet, as the initial effect of the rainfall excess will still be in transit. Summation over the area that is contributing gives us the following expression for the instantaneous unit hydrograph:

$$u(0, t) = \sum_{A(0)}^{A(t)} \frac{1}{\Pi(1+K_i D)} \cdot V_0 \delta(t-\tau) \cdot i \cdot \frac{\Delta A}{A}$$

If the integral is taken in the Stieltjes sense to allow for discontinuities, we can write this as

$$u(0, t) = \frac{V_0}{A} \int_0^{A(t)} \frac{\delta(t - \tau)}{\Pi(1 + K_i D)} . i . dA$$

in which

$u(0, t)$ = ordinate of the instantaneous unit hydrograph
V_0 = volume of runoff
A = area of catchment
t = time elapsed
τ = translation time
$K_1 \cdots K_n$ = storage delay times
$i(A)$ = ratio of local rainfall intensity to average over catchment.

The above equation is the general equation for the unit hydrograph of an ideal linear catchment in which translatory action and reservoir action are separated. If the rainfall intensity is constant over the catchment, then $i = 1$ and can be omitted. It is reasonable to suppose that any catchment suitable for unit hydrograph analysis can be represented by an equivalent ideal linear catchment consisting of linear channels and linear reservoirs so arranged as to give the same unit hydrograph as the natural catchment to the required degree of accuracy. Accordingly, the above equation is proposed as the general equation of the instantaneous unit hydrograph.

Methods of solution

The exact solution of the general equation derived in the last section would be a tedious business. It would require the evaluation for each element of the catchment area of the n-term series (generated by the n reservoirs lying between the element and the outlet) for all the values of $(t - \tau)$, and then the combination of the contributions from each element to obtain the complete unit hydrograph. From a practical viewpoint, the accuracy of the basic data renders the tedium of such a computation pointless, and in any case the accuracy required in applied hydrology makes this tedium unnecessary. Since the instantaneous unit hydrograph, which is itself an integral, is further integrated to obtain the actual hydrograph, small differences in catchment characteristics will not be reflected in the hydrograph of surface runoff due to the double smoothing involved. This suggests that the general equation of the unit hydrograph can be made tractable by the use of carefully chosen physical assumptions without undue loss of accuracy.

It is obviously desirable to transform the general equation from a surface integral to a single integral. This can be done as follows. For the purpose of solving the general equation, any point in the catchment is characterized by its translation time τ from the outlet and by the set of reservoirs with delay times $K_1 \cdots K_n$ lying between the point in question and the outlet. Imagine a 'contour' joining all points in the catchment which are separated from the outlet by the same translation time; such a line is termed an isochrone. Such isochrones cannot cross one another, cannot close, and can only originate or terminate on the boundary of the catchment. In general, it is possible for the points on an individual isochrone to have different characteristic 'reservoir chains' lying between the point and the outlet. This is illustrated for a simple case in Figure 1. Natural

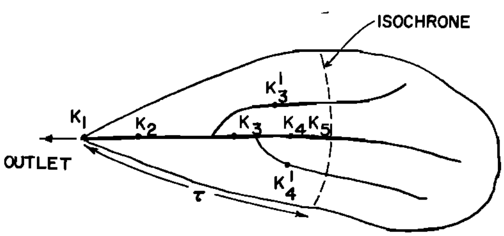

FIG. 1—General distribution of reservoirs

catchments, however, seem to conform to certain laws of equilibrium [*Horton*, 1945]. Thus the assumption that all points on the same isochrone have the same reservoir chain would not be unduly restrictive. This assumption is shown in Figure 2; it is clear that all the tributaries can be folded onto the main river to give a single chain of reservoirs as shown in Figure 3, the inflow at any point being proportional to the length of the isochrone ($dA/d\tau$) cutting the main river at that point. In order to take advantage of this assumption for non-uniform rainfall distribution, it is necessary to average the rainfall intensity along each isochrone, so that the intensity varies only with distance from the outlet.

The physical assumption made in the last

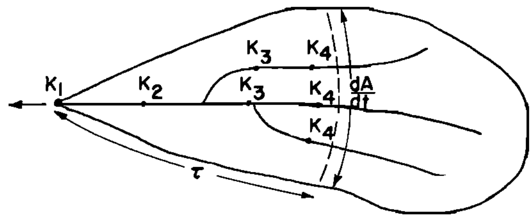

FIG. 2—Uniform distribution of reservoirs

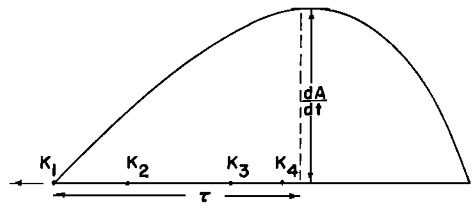

FIG. 3—Folding of tributaries onto main river

paragraph modifies the general equation

$$u(0, t) = \frac{V_0}{A} \int_0^{A(t)} \frac{\delta(t - \tau)}{\Pi(1 + K_i D)} . i . dA$$

to the following

$$u(0, t) = \frac{V_0}{A} \int_0^{t \le T} \frac{\delta(t - \tau)}{\Pi(1 + K_i D)} \, i \, \frac{dA}{d\tau} \, d\tau$$

where

$dA/d\tau$ = the length of the isochrone (i.e., the ordinate of the time-area curve)

T = the classical "time of concentration" (i.e., the maximum translation time in the catchment).

This can be written

$$u(0, t) = \frac{V_0}{T} \int_0^{t \le T} \frac{\delta(t - \tau)}{\Pi(1 + K_i D)} \, \omega\left(\frac{t}{T}\right) d\tau$$

where

$\omega\left(\frac{\tau}{T}\right)$ = ordinate of the dimensionless time-area concentration curve, adjusted for variation in rainfall intensity

or in compact dimensionless form

$$\frac{uT}{V_0} = \int_0^{t' \le 1} \frac{\delta(t' - \tau')}{\Pi(1 + K_i D)} \, \omega(\tau') \, d\tau'$$

which is a single integral since the chain of reservoirs is defined uniquely for each value of τ. It should be kept in mind that translation may include distributed storage in the form of continuous overbank flow. Hence in the present theory the translation time is based on the velocity of flow from the whole cross section and does not exclude overbank flow from the calculation. This is in contrast with the classical rational method, where time of concentration is based on bankfull flow conditions.

The integral given above can be further simplified by assumptions in relation to the time-area-concentration curve. A complex ω-curve can be broken down into simpler elements for which solutions are easily found, and then these solutions can be recombined to give the complex solution for the complete curve. In particular, if the curve is replaced by a series of straight lines each of which can be expressed as

$$\omega(\tau') = C_1 + C_2\tau' \qquad a < \tau < b$$

then the unit hydrograph is given by

$$\frac{uT}{V_0} = C_1 \int_a^b \frac{\delta(t - \tau)}{\Pi(1 + K_i D)}\, d\tau + C_2 \int_a^b \frac{\delta(t - \tau)}{\Pi(1 + K_i D)} \frac{\tau}{T}\, d\tau$$

In this way the unit hydrograph for any polygonal $\omega(\tau')$ can be built up from the basic unit hydrographs for $\omega = 1$ and $\omega = \tau'$. In many cases it will be simpler to solve the complex case by numerical integrations rather than by combining the basic solutions over short segments.

In order to compute an actual unit hydrograph from the general equation, the size and distribution of the linear reservoirs must be known or assumed. In general, these reservoirs may be of any size and placed at any given translation time from the outlet. The Zoch and Clark methods assume, in effect, that there is a single reservoir in the catchment. Using this assumption, the unit hydrograph can be computed in less than half an hour, even for the case of a complex time-area-concentration curve. If more reservoirs are inserted in the idealized catchment, the computational work is increased and would become practically impossible for a large number of unequal reservoirs. If, however, the reservoirs are all taken as equal, there is a great simplification and the number of reservoirs can be increased without extra work up to the limit of the tables used. In the case of a number of equal reservoirs, the computation takes two to three hours.

The solutions for these various assumptions regarding the linear reservoirs are discussed in greater detail in the remaining sections of the paper.

Existing solutions

The various theoretical solutions so far proposed for relating rainfall to runoff on a linear basis are easily derived as special cases of the general equation

$$\frac{uT}{V_0} = \int_0^t \frac{\delta(t - \tau)}{\Pi(1 + K_i D)} \cdot \omega(\tau') \cdot d\tau$$

The insertion of particular values in the above equation gives the solutions proposed in the Rational Method (1851), *Zoch* [1934], *Clark* [1945], and *Nash* [1957].

The Rational Method first proposed by Mulvany in 1851 [*Dooge*, 1957] treats the problem of runoff as one of translation only. This assumption corresponds to

$$K_1 = K_2 = \cdots = K_n = 0$$

which gives

$$\frac{uT}{V_0} = \int_0^t \delta(t - \tau)\omega(\tau')\, d\tau = \omega(t')$$

so that the instantaneous unit hydrograph has the same shape as the time-area-concentration curve. In the original form of the Rational Method, the critical storm duration was taken as equal to T, the time of concentration of the catchment. In the modified rational method, the critical storm is taken so as to maximize the product of the rainfall intensity and the peak of the finite period unit hydrograph.

Zoch [1936] and *Clark* [1945] both assumed that the runoff was routed through a single linear reservoir. Zoch assumed the storage to occur in the soil, while Clark more correctly regarded it as channel storage. This assumption corresponds to

$$K_1 = K$$

$$K_2 = K_3 = \cdots = K_n = 0$$

which when inserted in the general equation gives

$$\frac{uT}{V_0} = \int_0^t \frac{\delta(t-\tau)}{1+KD}\omega(\tau')\,d\tau$$

$$= \int_0^t \frac{1}{K}e^{-(t-\tau)/K}\omega(\tau')\,d\tau$$

$$= \frac{e^{-t/K}}{K}\int_0^t e^{\tau/K}\omega(\tau')\,d\tau$$

The latter equation is identical with the solution obtained when a time-area-concentration curve $\omega(\tau')$ is routed through a reservoir of delay time K in accordance with Clark's method. In practice, such a routing can be carried out very rapidly by using the coefficient form of the Muskingum method to obtain the instantaneous unit hydrograph. The finite-period unit hydrograph can be readily obtained by taking means and entering these in a further column; if required, a distribution graph can be obtained by using a further column [*Dooge*, 1956].

The instantaneous unit hydrographs computed on the assumption of a single reservoir for catchments with rectangular and triangular $\omega(\tau)$ curves are shown in Figures 4 and 5. The range of K/T values used corresponds with that normally found in natural catchments. It is found that the variation of the peak of the unit

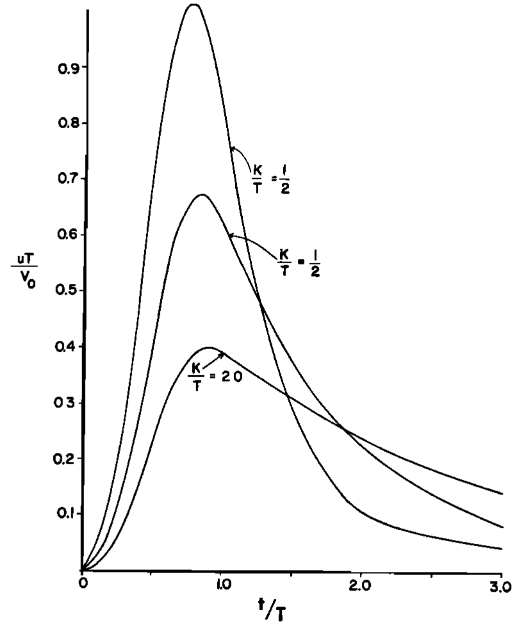

FIG. 5—Hydrograph for triangular catchment

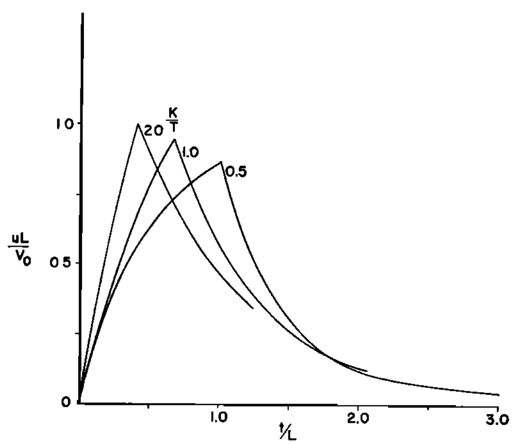

FIG. 6—Hydrograph in terms of lag (rectangular)

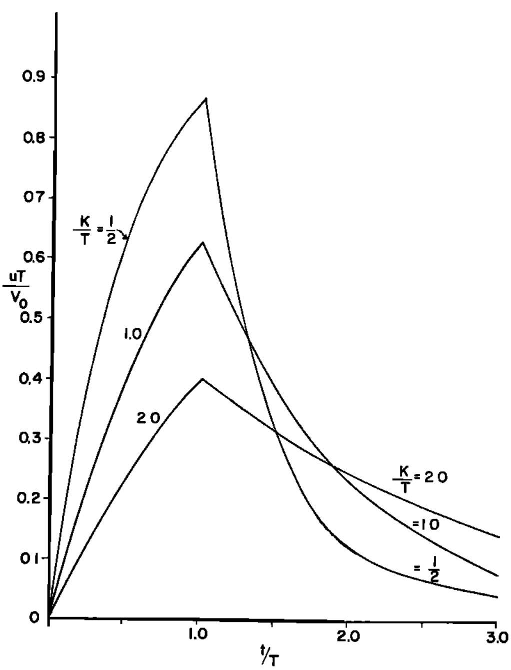

FIG. 4—Hydrograph for rectangular catchment

hydrograph is greatly reduced if the plotting is in terms of the lag (or first moment) L of the unit hydrograph. This plotting which is for uL/V_0 versus t/L is shown for rectangular and triangular catchments respectively in Figures 6 and 7. The remarkable constancy of the dimensionless peaks is shown in Table 1. It is clear from the above table that instantaneous unit hydrographs derived by the O.P.W. Method [*O'Kelly*, 1955], that is, by routing an isosceles triangle through a linear reservoir, will have the property that the product of the peak and

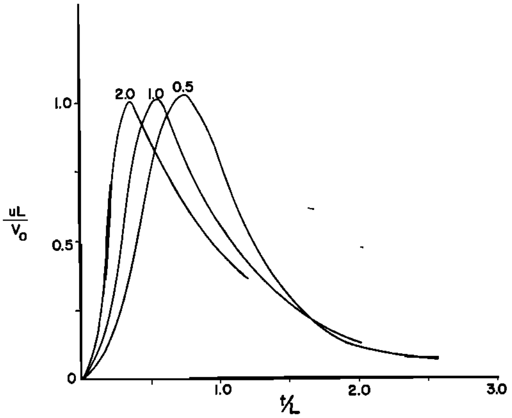

FIG. 7—Hydrograph in terms of lag (triangular)

the lag equal the volume within a few per cent.

Nash [1957] used a different approach in which he assumed that the operation of a catchment on rainfall excess to produce an instantaneous unit hydrograph was analogous to the operation of routing an instantaneous inflow through a series of equal linear reservoirs. The assumption of equal linear reservoirs transforms the general equation of the unit hydrograph as follows

$$\frac{uT}{V_0} = \int_0^t \frac{\delta(t-\tau)}{\Pi(1+K_iD)} \cdot \omega(\tau')\, d\tau$$
$$= \int_0^t \frac{\delta(t-\tau)}{(1+KD)^n} \cdot \omega(\tau')\, d\tau$$
$$= \int_0^t \frac{1}{K} \cdot \frac{m^{n-1}e^{-m}}{(n-1)!} \cdot \omega(\tau')\, d\tau$$

where

$$m = \frac{t-\tau}{K}$$

Since Nash ignores the variation in translation time over the catchment, he assumes in effect that all points have the same translation time; that is, that the dimensionless time-area-concentration curve is a δ-function. This gives

$$\frac{uT}{V_0} = \int_0^t \frac{1}{K} \frac{m^{n-1}e^{-m}}{(n-1)!}\, \delta(0)\, d\tau$$
$$\frac{uK}{V_0} = \frac{m^{n-1}e^{-m}}{(n-1)!} \qquad m = \frac{t}{K}$$

which is Nash's solution for the instantaneous unit hydrograph.

Nash's solution has the advantage that the

TABLE 1

K/T	$u_{max}L/V_0$	
	Rectangle	Triangle
.25	.74	1.03
.50	.87	1.02
1.0	.95	1.01
2.0	.98	1.00
3.0	.99	1.00
4.0	1.00	1.00

S-hydrograph is given by

$$S(t) = \int_0^t u(0, t) \cdot dt$$
$$= \frac{V_0}{K} \int_0^t \frac{m^{n-1}e^{-m}}{(n-1)!} \cdot dt$$
$$= V_0 \int_0^t \frac{m^{n-1}e^{-m}}{(n-1)!} \cdot dm$$
$$= V_0 \frac{\int_0^t m^{n-1}e^{-m} \cdot dm}{(n-1)!}$$
$$= V_0 \cdot I\left(\frac{t}{K}, n-1\right)$$

where $I(\ )$ is the ratio of the incomplete to the complete gamma function, which has been tabulated by *Pearson* [1930]. Thus the ordinates of the finite duration unit hydrograph can be found from the difference between two tabulated values

$$u(D, t) = \frac{S(t) - S(t-D)}{D}$$
$$= \frac{V_0}{D} \left\{ I\left(\frac{t}{K}, n-1\right) - I\left(\frac{t-D}{K}, n-1\right) \right\}$$

It is clear, therefore, that the solutions previously proposed as giving the form of the instantaneous unit hydrograph are special cases of the general equation derived earlier in the present paper. Even if the linear reservoirs are all assumed equal, the general method is extremely flexible, the result being dependent on the three numerical parameters T, K, and N, and on the functional parameters $\omega(\tau')$ and $n(\tau')$ defining the adjusted time-area-concentration curve and the reservoir

TABLE 2

Method	T	$\omega(\tau')$	K	N	$n(\tau')$	Degrees of freedom
Rational	variable	variable	0	0	—	2
Zoch-Clark	variable	variable	variable	1	at outlet	3
O'Kelly	variable	triangle	variable	1	at outlet	2
Nash	—	—	variable	variable	—	2
Using equal reservoirs	variable	variable	variable	variable	variable	5

distribution curve, respectively. The relation of the various special cases is shown in Table 2.

General solution using equal reservoirs

The assumption made in the Zoch-Clark method, that the reservoir storage in all parts of the catchment can be represented by a single linear reservoir at the outlet, seems unduly restrictive. Similarly, the neglect of translation in Nash's approach makes it impossible to analyze the effect on the unit hydrograph of such factors as the shape of the catchment and the ratio of overbank storage to channel storage. As indicated in the last section, the assumption that all the linear reservoirs in the idealized catchments are equal gives a solution having five degrees of freedom, compared with three and two degrees for the methods of Clark and Nash respectively. This additional flexibility is of great value, both for the direct analysis of the effect of catchment characteristics on the unit hydrograph, and also by providing additional parameters which can be used to improve the statistical analysis of derived unit hydrographs.

Any further generalization is probably not worthwhile. For the case of n unequal reservoirs, the solution of the general equation of the unit hydrograph is as follows

$$\frac{uT}{V_0} = \int_0^{t \le T} \frac{\delta(t-\tau)}{\Pi(1+K_i D)}\, \omega(\tau')\, d\tau$$

which gives

$$\frac{uT}{V_0} = \int_0^t \{C_1(\tau)e^{-(t-\tau)/K_1} + C_2(\tau)e^{-(t-\tau)/K_2} + \cdots + C_n(\tau)e^{-(t-\tau)/K_n}\}\cdot\omega(\tau')\cdot d\tau$$

The coefficients $C_1(\tau) \cdots C_n(\tau)$ are constant for the range corresponding to the reach between two adjacent reservoirs, but change on passing from downstream to upstream of each reservoir. Thus the computation is quite complex, becoming increasingly so as the number of reservoirs increases. Since this approach only gives more flexibility than the case of equal reservoirs if the number of unequal reservoirs is three or greater, the prospect is not encouraging. The case of unequal reservoirs is equally unpromising from the point of view of the numerical analysis of derived hydrographs. *Lanczos* [1957] has shown the uncertainty inherent in the analysis of exponential series. No advantage is obtained if two or more reservoirs are taken equal, since the number of terms remains the same and the individual terms are more complex. Simplification is only obtained if all the reservoirs are equal. From the practical viewpoint, the five degrees of freedom provided by the equal-reservoirs solution would appear to be ample for the purposes of both analysis and synthesis. In general, it may be concluded that the solution using equal reservoirs takes advantage of a considerable simplification without any appreciable loss of physical significance.

If all the linear reservoirs in the idealized catchment are taken equal in size, we have

$$\frac{uT}{V_0} = \int_0^t \frac{\delta(t-\tau)}{\Pi(1+K_i D)}\cdot\omega(\tau')\cdot d\tau$$

$$= \int_0^t \frac{\delta(t-\tau)}{(1+KD)^n}\cdot\omega(\tau')\cdot d\tau$$

$$= \int_0^{t/K} \frac{m^{n-1}e^{-m}}{(n-1)!}\cdot\omega(\tau')\cdot dm \qquad m = \frac{t-\tau}{K}$$

$$\frac{uT}{V_0} = \int_0^{t/K} P(m, n-1)\cdot\omega(\tau')\cdot dm$$

where

$$P(m, n-1)$$

is the Poisson probability function. Since the latter function has been tabulated by *Pearson*

[1930] and *Molina* [1942], the computation of any ordinate of the instantaneous unit hydrograph by numerical integration is comparatively easy. For any ordinate (i.e., for a fixed value of t) the variables under the integral sign can be evaluated for each value of τ, and the products integrated using Simpson's Rule, or some equivalent. For any value of τ, $\omega(\tau)$ is given by the time-area-concentration curve, n by the distribution $n(\tau)$ of the N reservoirs each of delay time K, and m by the difference $(t - \tau)$. A convenient tabulation is shown in Table 3 where $K = IT$ and $N = 20$.

TABLE 3

t'	τ'	m	$n(\tau')$	$P(m, n-1)$	$\omega(\tau')$	Δu
1.2	.00	12	1	.006	0	.0000
	.05	11.5	1	.010	.2	.0000
	.05	11.5	2	.116	.2	.0002
	.10	11.0	2	.184	.4	.0007

It is important to remenber that each reservoir introduces a change in the value of n and hence a discontinuity in the value $P(m, n - 1)$. For this reason it is necessary to evaluate the integrand immediately upstream and downstream of each reservoir.

APPLICATION OF PROPOSED SOLUTION

The detailed discussion of possible applications of the proposed equal-reservoirs equation of the unit hydrograph is beyond the scope of this paper. Nevertheless, some indication is desirable of the relevance of the solution to key problems in theoretical and applied hydrology. Even if the shapes of the time-area-concentration curve and reservoir-distribution curve are kept constant, the method has three degrees of freedom, each characterized by a single numerical parameter of physical significance. Variation of the shapes of the two curves mentioned gives a tremendous additional flexibility, if required, either in analysis or statistical correlation.

One obvious application is the use of the solution to predict the effect of various catchment characters on the shape of the unit hydrograph. In this respect the method can do all that Clark's method can do, and all that Nash's method can do, as might be expected, since each of these have been shown to be a special case of the equal-reservoirs solution. In addition, the proposed equation can analyze the effect on the unit hydrograph of the distribution of storage within the catchment. Such an analysis is of more than academic interest, since it could be used to determine the relative effects of the various catchment characteristics. In this way the optimum number of parameters to be used in correlation studies or recorded data could be determined in advance. It is interesting to note that the unit hydrograph will approach the shape of the time-area-concentration curve under three conditions: (a) as K approaches zero, (b) as n approaches infinity, and (c) as the distribution approaches the case of uniform distribution. Analysis could indicate the relative importance and sensitivity of these factors and could assess the extent to which any one of them could replace the others without serious error.

Special cases of the general solution can be handled algebraically with interesting results. An example of this is the case treated by O'Kelly [1955], that is, a triangular time-area curve, $n = 1$, placed at outlet and two variable parameters. In this case, as the present author pointed out in the discussion of O'Kelly, the peak of the instantaneous unit hydrograph is described below

$$\frac{u_{max} T}{V_0} = 4\left(1 - \frac{t_P}{T}\right)$$

$$e^{t_P/K} + 1 = 2e^{T/2K}$$

The latter equation can be expressed as

$$\frac{t_P}{K} = \log_e (2e^{T/2K} - 1)$$

which when expanded gives the following relationships within a few per cent

$$u_{max} L = V_0$$

$$\frac{t_P}{L} = 1 - \left(\frac{K}{L}\right)^2$$

The same approach can be made for other special cases. Thus for Nash's assumption of n equal reservoirs and no variation in time of translation, we have

$$\frac{uK}{V_0} = \frac{m^{n-1} e^{-m}}{(n-1)!} \qquad m = \frac{t}{K}$$

This is a maximum when

$$(n - 1)m^{n-2} - m^{n-1} = 0$$

i.e.

$$m_P = \frac{t_P}{K} = (n - 1)$$

or

$$\frac{t_P}{L} = \frac{t_P}{nK} = \frac{n-1}{n}$$

$$\frac{u_{\max}K}{V_0} = P(n - 1, n - 1)$$

The above result is exact.

For the case of a rectangular catchment using Clark's method we have

$$t_P = T$$

$$\frac{u_{\max}T}{V_0} = 1 - e^{-T/K}$$

which is also an exact result.

These three special solutions are plosted in Figure 8. It is a simple matter to plot the coordinates of a derived hydrograph on this diagram for comparison with the three special cases.

It has been pointed out by Nash (private communication of draft paper) that the moments of the unit hydrograph are suitable as a basis for correlations in unit hydrograph analysis. Though this may not be rigorously true [*Kendall*, 1943], it is probable that such an approach is

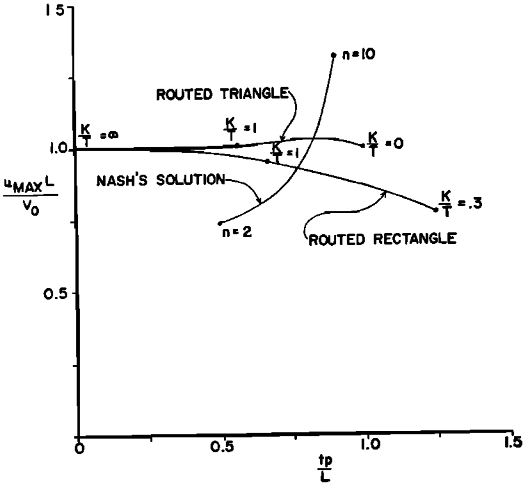

FIG. 8—Plotting of special solutions

the best available at the present time. It is desirable, therefore, to seek a convenient expression for the moments of the general equation

$$\frac{uT}{V_0} = \int_0^{t/K} \frac{m^{n-1}e^{-m}}{(n - 1)!}\cdot\omega(\tau')\cdot dm$$

which will be

$$M_R = \int_0^{\infty} dt\; t^R \int_0^{t/K} \frac{m^{n-1}e^{-m}}{(n - 1)!}\,\omega(\tau')\cdot dm$$

Integration in the above form would appear to be difficult. However, if the moment is constructed by taking the contribution of each element of the catchment, integrating with respect to time, and then integrating over the catchment, a solution becomes possible. In this form the moment is

$$M_R = \int_0^{T} d\tau\,\omega(\tau') \int_{\tau}^{\infty} \frac{m^{n-1}e^{-m}}{(n - 1)!}\,\frac{dt}{K}\cdot t^R$$

$$m = \frac{t - \tau}{K}; \quad t = mK + \tau; \quad dt = K\,dm$$

so that

$$M_R = \int_0^{T} d\tau\,\omega(\tau') \int_0^{\infty} \frac{m^{n-1}e^{-m}}{(n-1)!}\,(Km+\tau)^R\cdot dm$$

This can be evaluated by expanding the last term and taking advantage of the facts that τ is constant for the first integration and that

$$\int_0^{\infty} m^x e^{-m}\,dm = (x)!$$

Thus for the third moment we have

$$M_3 = \int_0^{T} d\tau\,\omega(\tau') \int_0^{\infty} \frac{m^{n-1}e^{-m}}{(n-1)!}\,(Km+\tau)^3\,dm$$

expanding $(Km + \tau)^3$ as $K^3m^3 + 3K^2m^2\tau + 3Km\tau^2 + \tau^3$ and integrating with respect to m, term by term

$$\int_0^{\infty} \frac{m^{n-1}e^{-m}}{(n - 1)!}\,K^3m^3\,dm = K^3(n + 2)(n + 1)n$$

$$\int_0^{\infty} \frac{m^{n-1}e^{-m}}{(n - 1)!}\,3K^2m^2\tau\,dm = 3K^2\tau(n + 1)n$$

$$\int_0^{\infty} \frac{m^{n-1}e^{-m}}{(n - 1)!}\,3Km\tau^2\,dm = 3K\tau^2\cdot n$$

$$\int_0^{\infty} \frac{m^{n-1}e^{-m}}{(n - 1)!}\,\tau^3\,dm = \tau^3$$

so that

$$M_3 = \int_0^T d\tau\, \omega(\tau')\{K^3(n+2)(n+1)n + 3K^2\tau(n+1)n + 3K\tau^2 n + \tau^3\}$$

which can be expressed as

$$M_3(u) = K^3\{M_0(n^3\omega) + 3M_0(n^2\omega) + 2M_0(n\omega)\} + 3K^2\{M_1(n^2\omega) + M_1(n\omega)\} + 3K\cdot M_2(n\omega) + M_3(\omega)$$

Thus the third moment of the unit hydrograph can be expressed in terms of the moments of the distributions defined by products of the time-area curve $\omega(\tau')$ and the reservoir distribution curve $n(\tau')$. The other moments can be similarly found.

Errors of approximation in proposed solution

The solution proposed is applicable to a catchment consisting only of linear channels and linear reservoirs, in which the reservoirs are all equal and are so placed that there is the same number of reservoirs between the outlet and all points in the catchment having the same translation time. In this final section some computations are made to determine the order of the approximations involved in these limitations on the complete generality of the solution.

The separation of translation and reservoir action is fundamental to the approach made in the present paper. The assumption that any linear reach can be replaced by a number of linear channels and linear reservoirs placed in series is analogous to the lumping of parameters used in the analysis of electrical networks. It is reasonable to make this assumption at least in the first formulation of the problem. The representation of uniformly distributed storage as a form of translation is also reasonable. In natural catchments such storage might be of two types: (a) continuous overbank storage, or (b) a large number of patches of overbank storage each of which has a delay time small in comparison with the 'time of concentration' of the catchment. In the case of continuous overbank storage, the fact of linearity implies that the velocity is constant for all flows under consideration. Thus the delay time due to overbank storage is included in the translation time. The presence in a reach of a length of overbank storage is reflected in the idealized catchment by a length of linear channel whose translation time per unit length is greater than that of the remainder of the reach. In the case of a succession of patches of overbank storage, the assumption that this can be replaced by an equivalent translation channel is exactly true only when the number of patches is infinite. However, it is a good approximation in many cases. Figure 9

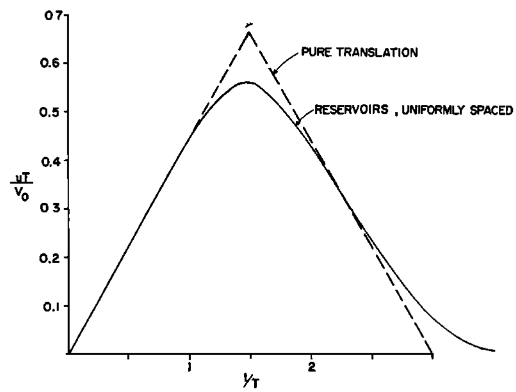

FIG 9—Translation approximated by twenty reservoirs

shows the comparison between the unit hydrograph for a triangular time-area curve with 20 reservoirs, each of delay time $T/10$, and the corresponding unit hydrograph for pure translation.

Once the assumption is made that translatory and reservoir action can be separated, the general equation of the unit hydrograph can be written as

$$u(0, t) = \int_0^{A(t)} \frac{\delta(t-\tau)}{\Pi(1 + K_i D)}\, i\, dA$$

and the remaining assumptions are made only to obtain a convenient solution. The first of the two assumptions used in the present paper to produce a convenient solution is that the chain of reservoirs $K_1 \cdots K_n$ appropriate to any point is determined completely by the total translation time between the point in question and the outlet. For this condition the disposition of reservoirs along a tributary and along the main channel upstream of a junction will be the same. Many natural catchments conform approximately to this condition of axial symmetry.

Computations were made to compare the total runoff due to each of three points of equal translation time, but with a different number of downstream reservoirs. Figure 10 shows the result for 8, 10, and 12 reservoirs, compared with the case for three branches each with 10 reservoirs. It will be seen from the figure that the case with the equal reservoirs gives a peak about 15 pct higher than that for the unequal reservoirs. If this difference occurs only over a small portion of the catchment, the corresponding error in the unit hydrograph will be much less than 15 pct. If differences occur over the greater part of the catchment, the error will be less than 15 pct in this case also, since the peak contributions from the different parts of the catchment will not coincide. We may conclude, therefore, that the above assumption, which transforms the surface integral of the general equation into an ordinary integral, will not give rise to serious error.

The remaining restriction on the generality of the proposed solution is the assumption that all the reservoirs are equal in size. Figure 11 shows the comparison between the outflow from a chain of three reservoirs of delay time K, and the outflow from a chain of reservoirs of delay time $0.5K$, K, and $1.5K$. Since the flows shown in Figure 11 are subjected to two smoothing processes (when integrating over the catchment and when passing to rainfall of finite duration),

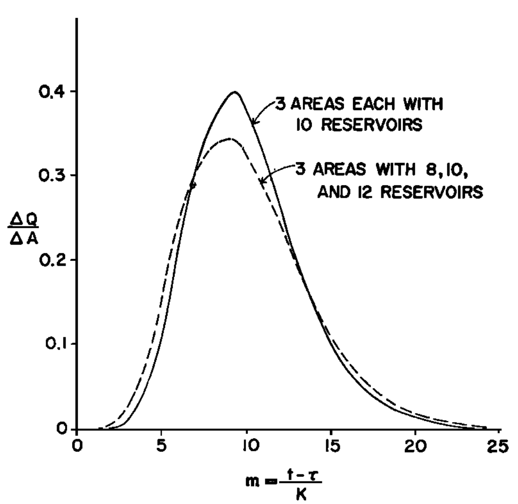

FIG. 10—Effect of unequal number of reservoirs

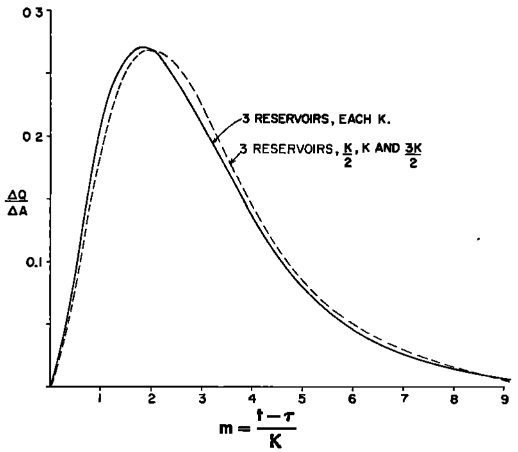

FIG. 11—Effect of unequal size of reservoirs

it will be obvious that the assumption of equal reservoirs will not be a source of appreciable error.

The trial computations made in this section show that for the cases examined the errors involved in the proposed solution are well within the normal accuracy of hydrological data. Accordingly, there is every reason to believe that the equation

$$\frac{uT}{V_0} = \int_0^t P(m, n-1)\omega(\tau')\, dm$$

$u = u(0, t)$ = ordinate of the instantaneous unit hydrograph
T = maximum translation time
V_0 = volume of rainfall excess
t = time elapsed since occurrence of rainfall excess
$P(m, n-1)$ = Poisson probability function
m = $(t-\tau)/K$ = dimensionless time factor
τ = translation time
K = size of linear reservoirs (all equal)
$n(\tau)$ = number of linear reservoirs downstream of τ
$\omega(\tau')$ = dimensionless time-area-concentration curve adjusted for variation in rainfall intensity

can be accepted as a flexible, convenient, and sufficiently accurate equation for the instantaneous unit hydrograph.

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References

Clark, C. O., Storage and the unit hydrograph, *Trans. Am. Soc. Civil Engrs.*, **110**, 1419–1446, 1945.

Dooge, J. C. I., Synthetic unit hydrographs based on triangular inflow, M. S. thesis, State University of Iowa, 1956.

Dooge, J. C. I., The rational method for estimating flood peaks—Irish contributions to the technique, *Engineering*, pp. 311–313, Sept. 6, 1957, pp. 374–377, Sept. 20, 1957.

Horton, R. E., Erosional development of streams and their drainage basins: hydrophysical approach to quantitative morphology, *Bull. Geol. Soc. Amer.*, **56**, 275–370, 1945.

Kendall, M. G., *The advanced theory of statistics*, 1, chap. 3 and 4, C. Griffin & Co., London, 1943.

Lanczos, C., *Applied analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1957.

Molina, E. C., *Poisson's exponential binomial limit*, D. Van Nostrand Co., New York, 1942.

Nash, J. E., The form of the instantaneous unit hydrograph, *C. R. et Rapports, Assn. Internat. Hydrol. IUGG*, Toronto, 1957.

O'Kelly, J. J., The employment of unit hydrographs to determine the flows of Irish arterial drainage channels, *J. Inst. Civil Engrs. (London)*, **4**, 365–445, 1955.

Pearson, K., Tables of the incomplete Γ function (1922) and Tables for statisticians and biometricians, *Biometrika*, University College, London, 1930.

Sherman, L. K., Streamflow from rainfall by the unit hydrograph method, *Eng. News-Record*, **108**, 501–505, 1932.

Zoch, R. T., On the relation between rainfall and stream flow, *Monthly Weather Rev.*, **62**, 315–322, 1934; **64**, 105–121, 1936; **65**, 135–147, 1937.

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