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A Semianalytical Solution for Partial Penetration in Two-Layer Aquifers

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A semianalytical solution is presented for the problem of drawdown distribution in a two-layer aquifer when the system is pumped from a well that is partially penetrating in one of the layers. The solution is used to illustrate the effects on aquifer behavior of partial penetration as well as the effect of a contrast in flow properties between the two layers. The validity of the solution has been verified against four available limiting cases. A method for analyzing field data is proposed, and an example is given to illustrate the procedure.

INTRODUCTION

Most aquifers in nature are more or less heterogeneous. A very common type of heterogeneity is found in stratified formations where the hydraulic properties of the porous media change from one layer to another. It is of great interest to predict the behavior of such aquifers when subjected to either withdrawal or injection operations.

Because of mathematical difficulties the analysis of transient fluid flow in multilayered aquifers has not received a great deal of attention. A rather simple case is that of a two-layer aquifer with no cross flow, i.e., the hydraulic connection between layers occurs only at the pumping well. *Lefkovits et al.* [1961] solved this problem for a bounded reservoir composed of two or more horizontal layers when the pumping well is fully penetrating and the rate of discharge is held constant. *Papadopulos* [1966] has studied the above case for two aquifers of infinite radial extent, and *Woods* [1970] has examined the same problem from the pulse test approach.

A more complex case of a layered aquifer occurs when the layers are hydraulically connected throughout their interface. *Katz* [1960] and *Russell and Prats* [1962], using different methods, have handled this problem for a bounded reservoir composed of two or more horizontal layers with a pumping well that is fully penetrating and a fluid level that is kept constant in the pumping well (constant terminal pressure). Because of a convergence problem, *Katz*'s [1960] solution does not lend itself to numerical evaluation when the radius of the well is less than 10 times the thickness of the aquifer, and consequently, it cannot be applied to groundwater problems.

A more practical case occurs when the rate of discharge rather than the water level is held constant. *Jacquard* [1960] has solved this problem when the pumping well penetrates the total thickness of the aquifer. So far, no numerical results have been obtained directly from his equations. *Pelissier and Sequier* [1961] have been able to invert the expression which Jacquard derived in the transform domain to obtain the pressure history at the well only. More recently, *Boulton and Streltsova* [1977*a*, *b*] have investigated the problem of flow in two-layer systems where one of the layers is fractured.

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In addition to the above analytical studies there have also been several numerical approaches to the layered aquifer problem. *Vacher and Cazbat* [1961] have used a finite difference method to obtain pressure distributions in a two-layer system with cross flow when a fully penetrating well is pumped at constant rate. *Javandel and Witherspoon* [1968*a*, 1969] applied the finite element method to solve problems of flow in multilayered aquifers.

It often happens that the pumping well does not penetrate or is not open over the whole thickness of the aquifer. The problem of partial penetration in a multilayered aquifer is one of the most complex to handle analytically. *Clegg and Mills* [1969] have considered a two-layer aquifer where both layers have finite thickness and the pumping well completely penetrates the top layer. They found that even for this special case the final solution could only be obtained when both layers had the same formation parameters. In effect this converts the problem into a single-layer, partial penetration problem that was solved much earlier by *Hantush* [1957].

Pizzi et al. [1965] used an electric analog model to study the effect of stratification on the performance of a well when it is only partially penetrating. This study revealed that the effect of stratification within the aquifer on the behavior of a partially penetrating well appeared to be like that of an extremely high, so-called 'apparent skin factor.' *Kazemi and Seth* [1969] have applied a finite difference technique to study the effect of anisotropy and stratification in a reservoir on pressure transient behavior of wells with restricted flow entry.

The above workers have been primarily interested in effects at the producing well because this is important in the field of petroleum engineering. In groundwater studies, however, one is often interested in the behavior of the aquifer away from the pumping well. *Boulton and Streltsova* [1975] have examined this problem for flow to a partially penetrating well that produces from an aquifer overlain by an aquitard.

In this paper we shall present a semianalytical solution for drawdown distribution in a two-layer aquifer drained by a well which partially penetrates only the top layer. The lower layer is considered to be very thick in relation to the upper layer. Since evaluation of the final solution is quite difficult, numerical inversion of the Laplace transformation has been

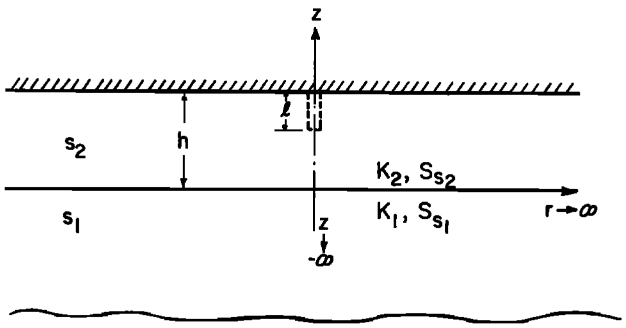

Fig. 1. Diagram of two-layer aquifer system with partially penetrating well.

applied to evaluate some practical cases and to study the effect of the parameters involved. A typical example where such a problem is commonly encountered involves relatively thin sands which overlay thick chalk in the London aquifer.

THEORY

Let us consider a mathematical model consisting of a system of a two-layered aquifer. As illustrated in Figure 1, each layer has its own flow properties and extends radially to infinity. The top layer has a finite thickness h and the lower one is relatively very thick, so that mathematically it behaves as a semi-infinite medium. It is assumed that both layers remain saturated throughout the period of investigation. It is also assumed that the initial drawdown is zero throughout the system. We require that the interface between the two layers have a perfect hydraulic contact and the upper boundary of the top layer be impermeable.

A well with infinitesimal radius has been placed in the top layer and is open along the length l from the top of the aquifer. This well will be pumped at a constant rate Q over a period of time t. The problem is to determine the drawdowns at any point of the system as a function of time.

The differential equation and initial and boundary conditions for this problem can be written as

$$\nabla^2 s_i = \frac{1}{\alpha_i}\frac{\partial s_i}{\partial t} \qquad i = 1, 2 \tag{1}$$

$$s_i(r, z, 0) = 0 \tag{2}$$

$$s_1(r, 0, t) = s_2(r, 0, t) \tag{3}$$

$$K_1 \frac{\partial s_1}{\partial z} = K_2 \frac{\partial s_2}{\partial z} \qquad z = 0 \tag{4}$$

$$\frac{\partial s_2}{\partial z}(r, h, t) = 0 \tag{5}$$

$$\lim_{r\to\infty} s_i(r, z, t) = 0 \tag{6}$$

$$\lim_{z\to-\infty} s_1(r, z, t) = 0 \tag{7}$$

$$\lim_{r\to 0} 2\pi K_2 r \int_{h-l}^{h} \frac{\partial s_2}{\partial r}\, dz = -Q \tag{8}$$

The $\bar{s}_1$ and $\bar{s}_2$ as given below represent the drawdowns in the Laplace transform domain due to a continuous point sink with unit strength at the point ($z = z_0$, and $r = 0$), in layers 1 and 2, respectively.

$$\bar{s}_1(r, z, \eta) = \int_0^\infty \frac{\xi J_0(\xi r) M}{2\pi\eta\alpha_2} \frac{\exp(\beta z - \gamma z_0)}{M\gamma + \beta}\, d\xi \tag{9}$$

$$\bar{s}_2(r, z, \eta) = \int_0^\infty \frac{\xi J_0(\xi r)}{4\pi\eta\alpha_2}\left\{\exp -[(z - z_0)\gamma] + \frac{M\gamma - \beta}{M\gamma + \beta}\exp -[(z + z_0)\gamma]\right\} d\xi \tag{10}$$

The $\bar{s}_1$ and $\bar{s}_2$, as given above, satisfy (1)-(4) as well as (6) and (7) in the Laplace transformed domain [*Javandel and Witherspoon*, 1968*b*]. An examination of the above two equations reveals that if we consider the whole system to have the properties of layer 2, drawdown in the top layer of this system is due to a sink of unit strength at the point $z = z_0$ as well as a sink at the point $z = -z_0$ but with a strength of $(M\gamma - \beta)/(M\gamma + \beta)$. Drawdown in the lower layer is due to a sink at $z = z_0/\sqrt{D}$ of strength $2M\beta/(M\gamma + \beta)$. Since $\bar{s}_1$ is expressed in the transformed domain in (9), the apparent location of this sink is at $z = (\gamma/\beta)z_0$. In this latter equation the whole system has the properties of the lower layer. One can now introduce the well-known method of images to satisfy the existence of the no-flow boundary at $z = h$. As a result, if we now set $A = (M\gamma - \beta)/(M\gamma + \beta)$, the following two equations are obtained which will also satisfy condition (5):

$$\bar{s}_1(r, z, \eta) = \int_0^\infty \frac{\xi J_0(\xi r)}{4\pi\eta\alpha_2\gamma}(1 + A)\left\{\exp\left[-\beta\left(\frac{\gamma}{\beta}z_0 - z\right)\right] + \exp\left[-\beta\left((2h - z_0)\frac{\gamma}{\beta} - z\right)\right] + \sum_{n=1}^{\infty} A^n\left[\exp\left[-\beta\left((2nh + z_0)\frac{\gamma}{\beta} - z\right)\right] + \exp\left[-\beta((2n + 2)h - z_0)\frac{\gamma}{\beta} + z\beta\right]\right]\right\} d\xi \tag{11}$$

$$\bar{s}_2(r, z, \eta) = \int_0^\infty \frac{\xi J_0(\xi r)}{4\pi\eta\alpha_2\gamma}\left\{\exp[-\gamma|z - z_0|] + \sum_{n=1}^{\infty} A^n[\exp[\gamma(z - (2nh + z_0))] + \exp[-\gamma(z + (2nh - z_0))]] + \sum_{n=0}^{\infty} A^n[A\exp[-\gamma(z + (2nh + z_0))] + \exp[\gamma(z + z_0 - h(2n + 2))]]\right\} d\xi \tag{12}$$

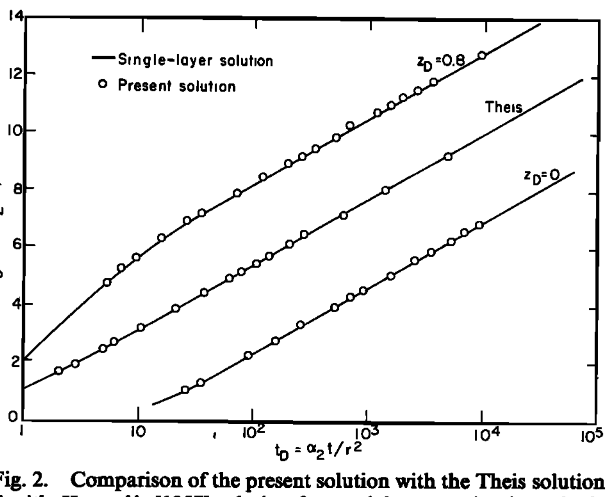

Fig. 2. Comparison of the present solution with the Theis solution and with *Hantush*'s [1957] solution for partial penetration in a single layer.

Integrating (11) and (12) with respect to z_0 from $h - l$ to h and adjusting for the strength of the sink lead to the following equations, which represent drawdown distribution due to a well of infinitesimal radius operating at constant rate:

$$\bar{s}_1(r, z, \eta) = \int_0^\infty \frac{Q\xi J_0(\xi r)(1 + A)}{4\pi l K_2 \eta \gamma^2} e^{\beta z} \cdot \left\{ \exp[-\gamma(h - l)] - \exp[-\gamma(h + l)] + \sum_{n=1}^{\infty} A^n \left[\exp[-\gamma((2n + 1)h - l)] - \exp[-\gamma((2n + 1)h + l)] \right] \right\} d\xi \quad (13)$$

$$\bar{s}_2(r, z, \eta) = \int_0^\infty \frac{Q\xi J_0(\xi r)}{4\pi l K_2 \eta \gamma^2} \{\exp[-\gamma(h - l - z)] - \exp[-\gamma(h + l - z)] + f\} d\xi \qquad z < h - l \quad (14)$$

$$\bar{s}_2(r, z, \eta) = \int_0^\infty \frac{Q\xi J_0(\xi r)}{4\pi l K_2 \eta \gamma^2} \left\{ 2 - \exp[\gamma(h - l - z)] - \exp[-\gamma(h + l - z)] + f \right\} d\xi \qquad z > h - l \quad (15)$$

where

$$f = \sum_{n=1}^{\infty} A^n \{\exp[\gamma(z + l - h(2n + 1))] - \exp[\gamma(z - l - h(2n + 1))] + \exp[-\gamma(z - l + h(2n - 1))] - \exp[-\gamma(z + l + h(2n - 1))]\} \quad (16)$$

If we introduce the following dimensionless terms: $s_D = 4\pi K_2 hs/Q$, $t_D = \alpha_2 t/r^2$, $r_D = r/h$, $l_D = l/h$, and $z_D = z/h$, (13)–(16) can be written

$$\bar{s}_{D_1} = \frac{1}{l_D} \int_0^\infty \frac{\xi J_0(\xi r_D)}{\eta \gamma^2} (1 + A) e^{\beta z_D} \left\{ \exp[-\gamma(1 - l_D)] - \exp[-\gamma(1 + l_D)] + \sum_{n=1}^{\infty} A^n [\exp[-\gamma((2n + 1) - l_D)] - \exp[-\gamma((2n + 1) + l_D)]] \right\} d\xi \quad (17)$$

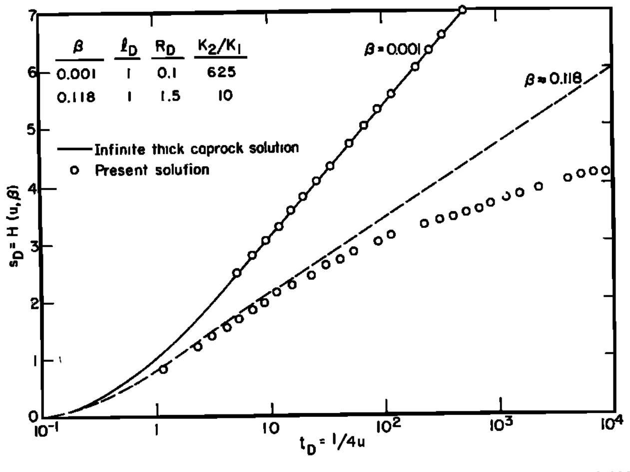

Fig. 3. Comparison of the present solution with *Hantush*'s [1960] solution for a leaky aquifer with an infinitely thick cap rock; β refers to a leakage parameter as defined by Hantush.

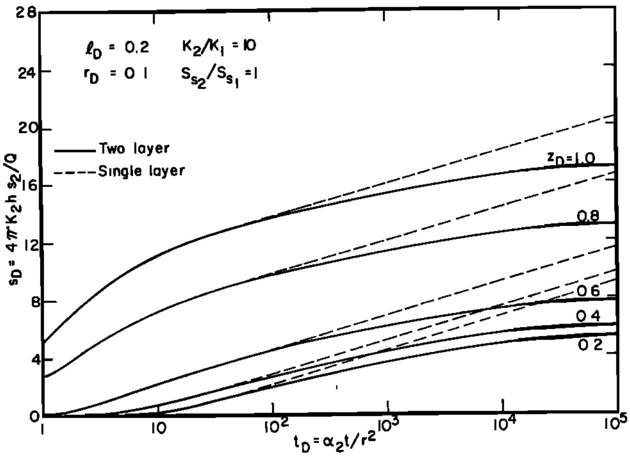

Fig. 4. Dimensionless drawdown versus dimensionless time at r_D = 0.1 for a two-layer aquifer compared with a single-layer system.

$$\bar{s}_{D_2} = \frac{1}{l_D} \int_0^\infty \frac{\xi J_0(\xi r_D)}{\eta \gamma^2} \{\exp[-\gamma(1 - l_D - z_D)] - \exp[-\gamma(1 + l_D - z_D)] + f_D\} d\xi \qquad z_D < 1 - l_D \quad (18)$$

$$\bar{s}_{D_2} = \frac{1}{l_D} \int_0^\infty \frac{\xi J_0(\xi r_D)}{\eta \gamma^2} \{2 - \exp[\gamma(1 - l_D - z_D)] - \exp[-\gamma(1 + l_D - z_D)] + f_D\} d\xi \qquad z_D > 1 - l_D \quad (19)$$

where

$$f_D = \sum_{n=1}^{\infty} A^n \{\exp[\gamma(z_D + l_D - (2n + 1))] - \exp[\gamma(z_D - l_D - (2n + 1))] + \exp[-\gamma(z_D - l_D + (2n - 1))] - \exp[-\gamma(z_D + l_D + (2n - 1))]\} \quad (20)$$

Analytical inversion of (18) and (19) is quite tedious, and once obtained, the results do not lend themselves easily to numerical evaluation. Therefore in order to draw meaningful re-

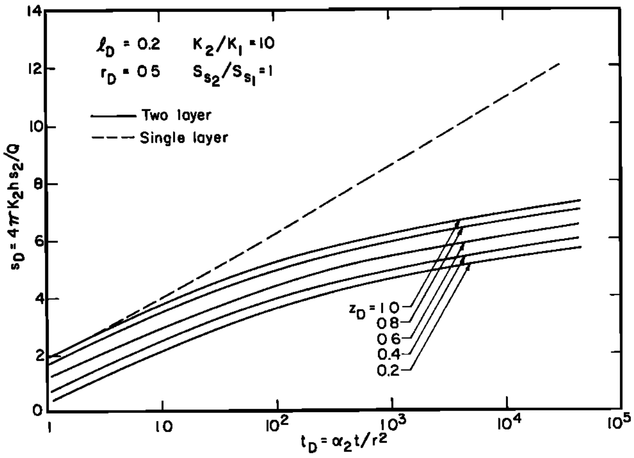

Fig. 5. Dimensionless drawdown versus dimensionless time at r_D = 0.5 for a two-layer aquifer compared with a single-layer system.

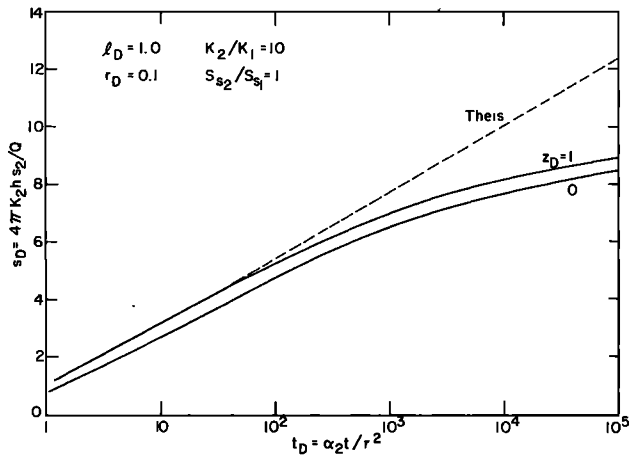

Fig. 6. Dimensionless drawdown versus dimensionless time in a two-layer system showing that flow is not necessarily horizontal in the more permeable layer.

sults from these equations one can apply numerical methods of inversion.

NUMERICAL INVERSION OF RESULTS

Numerical inversion of the Laplace transform has long been used for solving all kinds of engineering problems [*Bellman et al.*, 1966]. Several different methods are available which can be employed for numerical inversion, depending on the characteristics of the function to be inverted and the degree of accuracy that is required. A brief review of some common methods together with their application to groundwater problems is given elsewhere [*Javandel*, 1976].

Here, a method after *Bellman et al.* [1966] has been utilized for the inversion of $\bar{s}_D$. In this method, the inverse of $\bar{s}_D$ at a specified dimensionless time t_{D_i} may be obtained from the following formula:

$$s_D(t_{D_i}) = g(X_i) = \sum_{k=0}^{N-1} a_{ik}\bar{s}_D(k+1) \qquad i = 1, 2, 3, \cdots N \tag{21}$$

In the above equation, X_i are zeros of the shifted Legendre polynomial and

$$t_{D_i} = -\ln X_i \tag{22}$$

Extensive tables of the matrix a_{ik} are given by *Bellman et al.* [1966]. The zeros of the shifted Legendre polynomial are bounded between zero and unity, and thus one would expect to cover a time range of $(0, \infty]$. In practice, however, only a small range of time is obtained. In order to expand the range of t_D one may note that

$$L\{s_D(at_D)\} = \int_0^\infty e^{-\eta t_D} s_D(at_D)\, dt_D = \frac{\bar{s}_D(\eta/a)}{a} \tag{23}$$

Hence, if in (21) one uses $\bar{s}_D(1/a)/a$, $\bar{s}_D(2/a)/a$, $\cdots$, in place of $\bar{s}_D(1)$, $\bar{s}_D(2)$, $\cdots$, the values of t_D at which each numerical inversion is calculated would become

$$t_{D_i} = -a \ln X_i \tag{24}$$

Throughout this study, $N = 15$ has been used in (21). Since the reliability of this method rests on the accuracy of the calculation of $s_D(k + 1)$, integration of (17)–(19) has been performed by a 40-point Gauss-Laguerre quadrature formula. Elements of the series in (17) and (20) each represent the contribution of imaginary sinks above and below the top layer of the aquifer and therefore will vanish very rapidly when the sinks are at greater distances from the zone of interest.

At this point a short discussion about the stability of this procedure may be helpful. The unboundedness of the Laplace inverse operator is reflected in the behavior of the matrix a_{ik} in (21). As the dimension of this matrix increases, the magnitude of its elements, which have different signs, also increases. As a result, if one fails to calculate $\bar{s}_D$ with sufficient accuracy, the corresponding s_D calculated by (21) will contain large errors. However, in cases like ours, where $\bar{s}_d$ can be determined to any degree of accuracy, one can be confident of obtaining correspondingly accurate values of s_D [*Bellman et al.*, 1966].

VERIFICATION OF THE SOLUTION

Results obtained from (17)–(20) were verified against four limiting cases.

1. The present solution should converge to the Theis solution if we set the depth of penetration of the pumping well equal to the total thickness of the top layer and the permeability of the lower layer vanishes. This has been checked analytically by letting $l_D = 1$ and $A = 1$ in (18) and (19). It has also been checked directly by letting $l_D = A = 1$ in the program, and the results are shown on Figure 2. Agreement between the limiting case of the present solution and the Theis solution for a single layer is excellent.

2. When the permeability of the lower layer is set to zero, the solution should match the case of *Hantush*'s [1957] solution for partial penetration in a single layer. This will lead to $A = 1$, and Figure 2 includes a comparison of our solution with the single-layer solution for $l_D = 0.5$, $r_D = 0.1$, and $z_D = 0.8$ and 0.0.

3. If flow properties of both layers are identical, then the solution should merge to the one given by *Saad* [1960] for a thick artesian aquifer. This can easily be verified by letting $M = D = 1$, which will lead to $A = 0$.

4. When the pumping well penetrates all the way through the top aquifer and $K_1 \ll K_2$, one would expect that, at least at early time, our solution should agree with the leaky aquifer theory of *Hantush* [1960] for an infinitely thick cap rock. We examined this by letting $r_D = 0.1$, $K_2/K_1 = 625$, and $S_{s_2}/S_{s_1} = 1$; Hantush's β parameter can then be computed from

$$\beta = \frac{r}{4h}\left(\frac{K_1 S_{s_1}}{K_2 S_{s_2}}\right)^{1/2} = \frac{0.1}{4}\left(\frac{1}{625}\right)^{1/2} = 0.001$$

Figure 3 shows the good agreement between Hantush's solution and ours for $\beta = 0.001$. As the β parameter of Hantush increases, however, this agreement will only occur at early time. To demonstrate this, we set $r_D = 1.5$, $K_2/K_1 = 10$, and $S_{s_2}/S_{s_1} = 1$ for which $\beta = 0.118$. Figure 3 shows how results from the two solutions deviate as $t_D > 1$. These differences are to be expected because Hantush assumed vertical flow in the confining layer, and this will not hold when K_2/K_1 is as small as 10. Our new solution can thus be used to determine limiting conditions for the applicability of *Hantush*'s [1960] leaky aquifer theory and the subsequent work on this problem by *Neuman and Witherspoon* [1969*a*, *b*].

DISCUSSION OF RESULTS

From (17)–(19) we note that in order to investigate the variation of drawdown with time we must consider the effects of five parameters: l_D, r_D, z_D, M, and D. It is not practical to attempt to tabulate solutions to these equations here, but an extensive table covering a wide range in the parameters is being prepared as a separate report. A limited number of results will be presented in the form of graphs to illustrate some important points.

At small values of time, drawdown in the aquifer (layer 2) is similar to that of the single-layer partial penetration problem. This can be seen on Figure 4 where curves for z_D from 0.2 to 1.0 all coincide with the single layer results for $t_D < 50$. At longer times, when the contribution of the lower layer becomes significant, the amount of drawdown drops below the corresponding value for a single-layer partial penetration problem. At larger values of z_D the effect of the lower layer is sensed at a later time, which in effect causes a larger value for departure time. These results were obtained for $r_D = 0.1$, and Figure 5 shows the effect of increasing to $r_D = 0.5$. To avoid crowding the figure, the solution of the single-layer partial penetration problem has been shown only for $z_D = 1$. At greater distance from the pumping well the family of nondimensional curves for drawdown is more compact (note that the vertical scale on Figure 5 has been enlarged by a factor of 2). This indicates that the effect of partial penetration diminishes with distance from the pumping well. One may also note that the time of departure of the single layer from the two-layer solution has been decreased to a t_D about 25 times smaller than that for the case of $r_D = 0.1$. In fact, an approximate formula for departure time may be given as

$$(t_D)_d \approx \frac{(1 - l_D + z_D)^2}{5{r_D}^2} \qquad (25)$$

When $l_D = 1$, which means the pumping well is open all the way through the top layer, the solution applies to the case of an aquifer which is overlain by a relatively thick, leaky confining layer. This solution does not have the restriction that is considered in almost all available solutions for leaky aquifers where flow is assumed to be only vertical in the aquitard and only horizontal in the aquifer itself. Figure 6 shows the variation of dimensionless drawdown versus dimensionless time at the top and bottom of the top layer, where $K_2/K_1 = 10$ and $r_D = 0.1$. The Theis curve has also been shown for reference purposes. This figure shows that in fact, equipotentials are not always vertical in the aquifer.

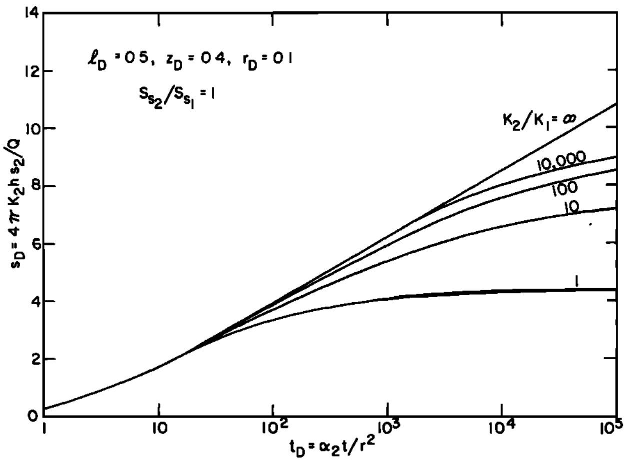

Fig. 8. Effect of permeability contrast on drawdowns in the more permeable layer of a two-layer system.

In the case of single-layer partial penetration it was observed that at relatively large distances from the pumping well (r greater than 1.5 times the thickness of the aquifer), the effect of partial penetration vanishes and the aquifer behaves as if the pumping well were fully penetrating [*Hantush*, 1957; *Javandel and Witherspoon*, 1967]. The same phenomena is observed in these results except that the effect of leakage from the lower layer will still be manifested. Figure 7 illustrates this for a partial penetration of $l_D = 0.2$ at $r_D = 1.5$; the corresponding curves essentially coincide with the case of full penetration for $l_D = 1.0$, $r_D = 1.5$.

As mentioned above, at early time after the start of pumping, the response of the aquifer is as if the lower layer were absent. However, at later times the behavior is completely different, and the amount of deviation from the single-layer case depends on the contrast in permeability between the two layers. Figure 8 shows the effect of permeability contrast for the case of $l_D = 0.5$, $z_D = 0.4$, and $r_D = 0.1$. In order to illustrate the effectiveness of the lower layer in terms of leakage, Figure 9 has been prepared for the same parameters as that of Figure 8. Figure 9 shows the difference between drawdowns in the single layer and that of the two-layer case in relation to the single-layer solution. At any given time the area under each curve indicates the percent of the total volume of fluid pro-

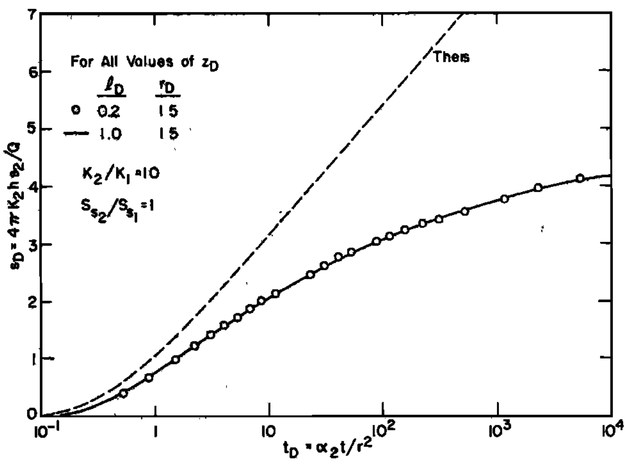

Fig. 7. Dimensionless drawdown versus dimensionless time showing that effects of partial penetration disappear in two-layer aquifers as $r_D \geq 1.5$.

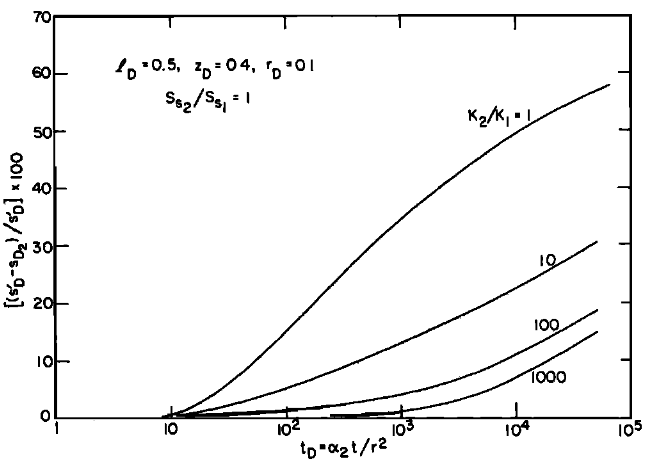

Fig. 9. Percent drawdown in upper (permeable) layer due to leakage from lower layer at $r_D = 0.1$.

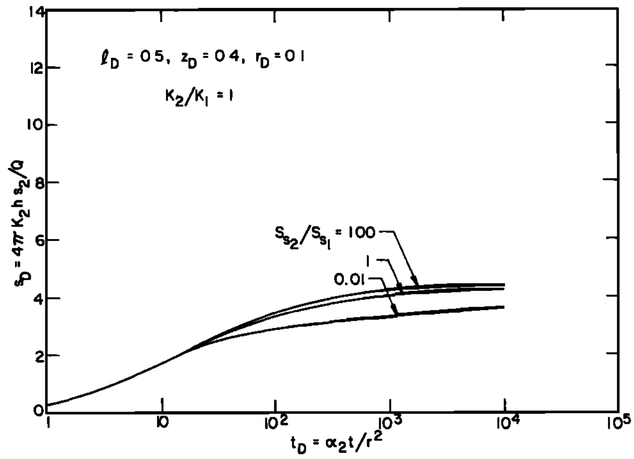

Fig. 10. Effect of specific storage contrast on dimensionless drawdown versus dimensionless time in a two-layer system.

duced that has been drawn from the lower layer at $r_D = 0.1$.

One may note that for the sake of convenience the ratio of specific storage in the two layers has been assumed to be unity in all of the above examples. Other storage values are easily investigated without introducing any complexity. For example, Figure 10 shows the effect of varying the contrast in specific storage from 0.01 to 100, which goes well beyond the limits usually observed in the field, for the particular case of $l_D = 0.5$, $z_D = 0.4$, $r_D = 0.1$ and $K_2/K_1 = 1$. A comparison of results between Figures 8 and 10 indicates that the effect of varying the contrast in specific storage has much less effect than variations in the permeability contrast. This can also be seen from (18)–(20). The term corresponding to the contrast in specific storage appears only in f_D, where its total effect compared to the first two terms of the integrand is of much smaller magnitude.

Interpretation of Field Data

As discussed above, drawdown versus time is a function of l_D, r_D, z_D, M, and D. To assist in the interpretation of field data, it was found that the problem is greatly simplified if observation wells are provided with the same depth and amount of penetration as the pumping well. There are two advantages in such a procedure. First, it is usually simpler to construct an observation well over some part of the aquifer than to install a piezometer. Second, the solution for drawdown in such an observation well is much simpler than that for a piezometer (as given by (17)–(20)). This can be demonstrated by integrating (19) with respect to z_D and dividing the result by the length of the observation well, which in this case is l_D. In the Laplace transform domain the solution takes the following form in which drawdown versus time is now only a function of l_D, r_D, and M:

$$\bar{s}_{D_2} = \frac{1}{l_D^2} \int_0^\infty \frac{\xi J_0(\xi r_D)}{\eta \gamma^2} \left\{ 2l_D - \frac{1}{\gamma} + \frac{e^{-2\gamma l_D}}{\gamma} + \frac{\sinh^2(\gamma l_D)}{\gamma} \sum_{n=1}^{\infty} 4A^n e^{-2n\gamma} \right\} d\xi \qquad (26)$$

The results on Figures 4 and 5 show that the drawdown behavior at early time occurs as though there were only a single layer. Therefore the properties of the single (upper) layer can readily be established using conventional methods. An analysis of the results indicates that in setting up a pumping test one should select the location of observation wells so that $r_D < 0.5$. One should also note from (25) that for a given l_D and z_D the closer the observation well is to the pumping well, the longer the single-layer response will last.

After determining the results for the upper (pumped) layer the variations from the single-layer response at later times can be used to determine the properties of the lower (unpumped) layer. To do this, one can choose the parameters l_D and r_D and calculate the variation of dimensionless drawdown as a function of dimensionless time for various ratios of K_2/K_1. The difference between these values of drawdown and the corresponding drawdowns for a single-layer solution, which we shall call Δs_D, can then be plotted on semilog paper for different values of K_2/K_1. Figure 11 shows such a family of curves for $l_D = r_D = 0.2$. When other values of r_D are needed, the same families of curves will result for the same value of l_D except that there must be a shift in the time axis. As a result, the curves shown on Figure 11 are independent of r_D and when used together with type curves for a single-layer partial penetration case can be employed to interpret field data for $l_D = 0.2$. It will be necessary to generate different families of curves for values of l_D other than 0.2.

The following example will illustrate the procedure to be used in interpreting field data. Table 1 shows results for drawdown versus time from a hypothetical field test where the aquifer consists of two layers. The top layer is 40 m thick and the bottom layer is very thick by comparison. Both the pumping and observation wells are completed in the top 8 m of the upper layer, and the distance between the wells is 4 m. The rate of discharge is 0.02 m^3/s. The problem is to determine the flow properties in both layers of the aquifer.

The following procedure should be used in the interpretation of these data:

1. Prepare a log-log plot as shown in Figure 12 for the average dimensionless drawdown for an observation well with $l_D = 0.2$ versus dimensionless time for $l_D = 0.2$ and $r_D = 0.1$ from the single-layer solution of *Hantush* [1961].

2. Plot the data of Table 1 for drawdown versus time on another sheet of log-log paper with the same scale per log cycle.

3. Find a match point by superposing the two plots, being careful to use only early time data. The coordinates of the ar-

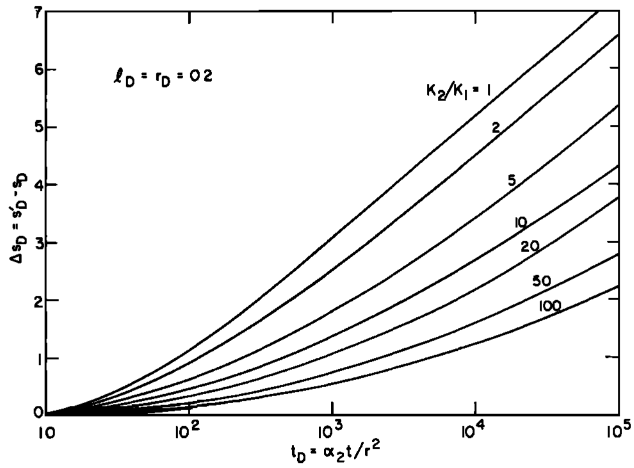

Fig. 11. Effect of permeability contrast on deviation in dimensionless drawdown for a two-layer aquifer from the result in a single-layer system.

bitrary match point chosen here are $t = 1600$ s for $t_D = 10$ and $s = 0.2$ m for $s_D = 1$.

4. From the definitions of t_D and s_D, find $K_2 = 0.0002$ m/s and $\alpha_2 = 0.1$ m²/s.
5. With the two curves superposed, read the values of $s_D = s_D' - s_D$ for several different values of t_D.
6. Plot the values of Δs_D versus t_D on semilog paper with the appropriate scale, and superpose, as on Figure 11.

It is necessary to shift the two curves in stage 6 parallel to the t_D axis in order to obtain the best match with the curves of K_2/K_1. In this way one can then estimate the value for the permeability ratio. From the data tabulated in Table 1 we obtain a result of $K_2/K_1 = 10$. At this point, two comments may be helpful. First, there must be an appropriately long period of pumping in order for the drawdowns to deviate from single-layer behavior. Second, after the properties of the top layer have been determined, it may prove more accurate to convert the pump test data into dimensionless results and subtract them from corresponding values of s_D' for the single-layer case. This should lead to a better result than will be obtained in attempting to determine Δs_D directly from the log-log results shown on Figure 12. Finally, one may note that the accuracy of this approach decreases as the ratio K_2/K_1 increases.

CONCLUSIONS

A semianalytical solution has been presented for the problem of drawdown distribution in a two-layer aquifer when water is pumped from a well that only partially penetrates one of the layers. The validity of the solution has been verified against four available limiting cases. Analysis of the results has revealed several important points. At small values of time, drawdown in the pumped layer is similar to that of the case of partial penetration in a single layer with the same properties. The effects of partial penetration disappear as distance exceeds 1.5 times the thickness of the pumped layer, very much the same as in the case of a single-layer aquifer. Available type curves for the standard case of partial penetration when combined with families of curves such as shown in Figure 11 can be used to determine the hydraulic properties of both layers. The contribution of water from each of the layers can be determined through an application of the type of curves illustrated in Figure 9.

TABLE 1. Drawdown Versus Time From Two-Layer Pumping Test

Time, min	Drawdown, m
2	0.70
3	0.94
5	1.22
10	1.58
15	1.75
20	1.85
40	2.07
60	2.18
90	2.26
120	2.30
180	2.24
360	2.50
720	2.60
1440	2.70
2160	2.79
2880	2.81
7200	2.91
14400	2.98
28800	3.03

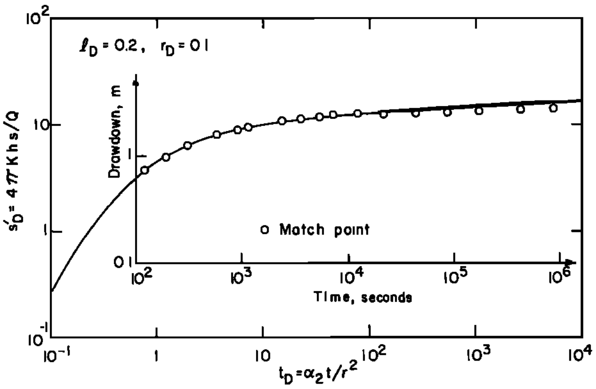

Fig. 12. Drawdown data for a two-layer system compared with corresponding single-layer-type curve.

NOTATION

$A = (M\gamma - \beta)/M\gamma + \beta)$.
$D = \alpha_2/\alpha_1$.
h thickness of the top layer, L.
$J_0(x)$ Bessel's function of the first kind and zero order.
K_1, K_2 permeability of layers 1 and 2, respectively, L/T.
l depth of penetration, L.
$l_D = l/h$.
$M = K_2/K_1$.
Q rate of discharge, L^3/T.
r radial distance, L.
$r_D = r/h$.
s_1, s_2 drawdown of layer 1 and 2, respectively, L.
$s_{D_i} = 4\pi K_2 h s_i/Q$.
S_{s_1}, S_{s_2} specific storage of layer 1 and 2, respectively, L^{-1}.
s_D' dimensionless drawdown for a single-layer aquifer.
t time, T.
$t_D = \alpha_2 t/r^2$.
z vertical coordinate, L.
$z_D = z/h$.
z_0 vertical coordinate of a point sink, L.
α_1, a_2 diffusivity of layer 1 and 2, respectively, L^2/T.
$\beta = (\xi^2 + \eta D)^{1/2}$.
$\gamma = (\xi^2 + \eta)^{1/2}$.
∇^2 Laplacian operator.
η Laplace transform parameter.
ξ Hankel transform parameter.

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