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Analytical Solution of a Partially Penetrating Well in a Two-Layer Aquifer

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The behavior of a two-layer aquifer pumped by a partially penetrating well is a matter of considerable interest. Analytical solutions are developed for drawdowns in either layer and are evaluated numerically to illustrate some typical cases. The validity of the solutions is demonstrated by comparison with the limiting single-layer case and with results from finite element calculations. Asymptotic solutions for small and large values of time are developed to show that (1) at early times with partial penetration the behavior of the pumped layer is exactly the same as that of a single layer and (2) at large values of time a semilog plot of drawdown versus time yields a straight line whose slope is only a function of the sum of the two transmissivities. A method is proposed for the interpretation of pump test data.

INTRODUCTION

The hydraulic response of a layered aquifer under the influence of a pumping well is a problem of interest in the fields of hydrogeology, geothermal engineering, and petroleum engineering. Numerous papers have been written on various aspects of this problem. *Hantush and Jacob* [1955] have presented solutions for steady state flow to a well draining one of the layers of a two-layer bounded aquifer. *Lefkovits et al.* [1961] studied the transient performance of a stratified bounded reservoir where the producing well is completely penetrating and there is no crossflow. *Papadopulos* [1966] has studied the same problem for only two layers of infinite areal extent. A similar problem but with crossflow between adjacent layers has also been investigated by *Katz* [1960] and *Russell and Prats* [1962] for the case of constant head at the wellbore and by *Jacquard* [1960] for constant flow rate.

In addition to the above works, which are all based on the analytical approach, many authors have applied numerical as well as analog models to problems of flow in layered aquifers [*Vacher and Cazbat,* 1961; *Pizzi et al.,* 1965; *Javandel and Witherspoon,* 1968, 1969; *Neuman and Witherspoon,* 1969; *Kazemi and Seth,* 1969]. Recently, *Javandel and Witherspoon* [1980] studied the problem of flow to a partially penetrating well in a two-layer aquifer where the well is open in the top layer and the lower layer is considered to be infinitely thick.

In this paper we shall present an analytic solution to the problem of transient flow to a partially penetrating well that is open in either layer of a two-layer system where both layers are finite in thickness. Crossflow is permitted at the interface between the two layers. These solutions can easily be evaluated numerically. Asymptotic forms of the solution for small and large values of time are developed from the general solution. The approach here is to solve the problem when the pumping well is only partially open in the top of the upper layer. Note that by symmetry, this is the same as the case when the pumping well is only partially open in the bottom part of the lower layer.

A second solution is developed for the case when the pumping well is only partially open within the lower layer but just below the interface between the two layers. Just as before, by symmetry, this solution is also valid for the case when the pumping well is open within the upper layer but just above the interface.

The analytical solutions are evaluated numerically and results are presented in dimensionless form on semilogarithmic plots for a few different parameters. Based on the application of these results, a method is proposed for interpretation of the pump test data in two-layer aquifers.

WELL OPEN IN THE TOP LAYER

Let us consider an aquifer consisting of two layers that are confined above and below by impervious layers, as illustrated on Figure 1. Each layer has its own flow properties, is finite in thickness, and extends radially to infinity. The interface between the two layers is an open boundary, meaning that no discontinuity of potential or its gradient is allowed across this surface. As illustrated in part *a* of Figure 1, the top layer of the system is partially penetrated by a well of infinitesimal radius for a length l from the top of the aquifer. If the well is pumped at a constant rate Q, we are interested in determining the value of drawdown, $s(r, z, t)$, at any point in the aquifer after pumping starts. By symmetry, this solution will also apply to the configuration given in part *b* of Figure 1. Note that the origin is on the axis of the well at the interface between the two layers. The differential equations and initial and boundary conditions to describe this problem can be written as

$$\frac{\partial^2 s_i}{\partial r^2} + \frac{1}{r}\frac{\partial s_i}{\partial r} + \frac{\partial^2 s_i}{\partial z^2} = \frac{1}{\alpha_i}\frac{\partial s_i}{\partial t} \qquad i = 1, 2 \tag{1}$$

$$s_i(r, z, 0) = 0 \tag{2}$$

$$\frac{\partial s_1}{\partial z}(r, h_1, t) = 0 \tag{3}$$

$$\frac{\partial s_2}{\partial z}(r, -h_2, t) = 0 \tag{4}$$

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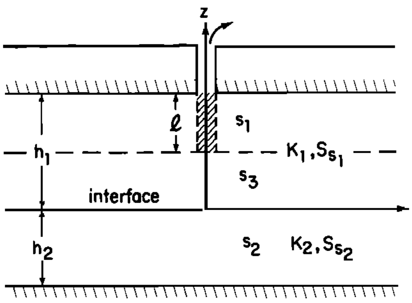

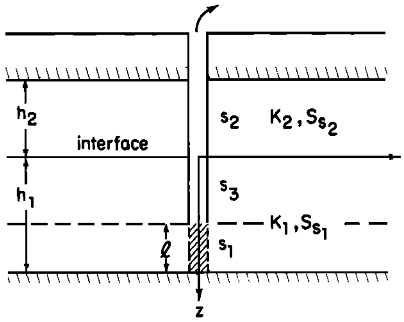

Fig. 1. Schematic diagrams of two-layer aquifers, (*a*) with a partially penetrating well in the top of the upper layer and (*b*) when the pumping well is only open in the bottom of the lower layer.

$$\lim_{r\to\infty} s_i(r, z, t) = 0 \tag{5}$$

$$s_1(r, 0, t) = s_2(r, 0, t) \tag{6}$$

$$K_1 \frac{\partial s_1}{\partial z}(r, 0, t) = K_2 \frac{\partial s_2}{\partial z}(r, 0, t) \tag{7}$$

$$\lim_{r\to 0}\left(r\frac{\partial s_1}{\partial r}\right) = -\frac{Q}{2\pi K_1 l} \qquad (h_1 - l) < z < h_1 \tag{8}$$

$$\lim_{r\to 0}\left(r\frac{\partial s_i}{\partial r}\right) = 0 \qquad -h_2 < z < (h_1 - l) \tag{9}$$

The above formulation is based on the assumption that both layers have an isotropic hydraulic conductivity. If, however, the radial hydraulic conductivity of each layer differs from that of the vertical, then by replacing r with $r(K_z/K_r)^{1/2}$ and considering $\alpha = K_z/S_s$ in the final solution, one obtains the answer for the anisotropic system. This procedure is only valid, however, when the ratio of K_z/K_r is the same in both layers.

In order to handle the nonuniform boundary condition along the axis of the well, one can arbitrarily divide the top layer of the aquifer into two separate layers by considering an imaginary interface at the elevation of $z = h_1 - l$. The system is then made of three layers, two of them having the same flow properties. Let us then designate three different symbols for drawdown: s_1 for the top layer in the zone between the top of the aquifer and the imaginary horizontal plane passing through the bottom of the well, s_2 for the bottom layer, and s_3 for the zone between the elevation of the bottom of the well and the top of the lower layer.

The solution of the problem can be obtained by successive application of Laplace and Hankel transformations over t and r, respectively. If we indicate the Laplace transform of $s_i(r, t)$ by $\bar{s}_i(r, p)$ and the Hankel transform of $\bar{s}_i(r, p)$ by $\bar{\bar{s}}_i(\xi, p)$, then (1) through (9) become

$$\frac{d^2\bar{\bar{s}}_1}{dz^2} - \omega_1^2\bar{\bar{s}}_1 = \frac{-Q}{2\pi l K_1 p} \qquad (h_1 - l) < z < h_1 \tag{10}$$

$$\frac{d^2\bar{\bar{s}}_2}{dz^2} - \omega_2^2\bar{\bar{s}}_2 = 0 \qquad -h_2 < z < 0 \tag{11}$$

$$\frac{d^2\bar{\bar{s}}_3}{dz^2} - \omega_1^2\bar{\bar{s}}_3 = 0 \qquad 0 < z < (h_1 - l) \tag{12}$$

$$\frac{d\bar{\bar{s}}_1}{dz}(\xi, h_1, p) = 0 \tag{13}$$

$$\frac{d\bar{\bar{s}}_2}{dz}(\xi, -h_2, p) = 0 \tag{14}$$

$$\bar{\bar{s}}_1(\xi, h_1 - l, p) = \bar{\bar{s}}_3(\xi, h_1 - l, p) \tag{15}$$

$$\bar{\bar{s}}_3(\xi, 0, p) = \bar{\bar{s}}_2(\xi, 0, p) \tag{16}$$

$$\frac{d\bar{\bar{s}}_1}{dz}(\xi, h_1 - l, p) = \frac{d\bar{\bar{s}}_3}{dz}(\xi, h_1 - l, p) \tag{17}$$

$$K_2\frac{d\bar{\bar{s}}_2}{dz}(\xi, 0, p) = K_1\frac{d\bar{\bar{s}}_3}{dz}(\xi, 0, p) \tag{18}$$

where

$$\omega_1 = \left(\frac{p}{\alpha_1} + \xi^2\right)^{1/2} \qquad \omega_2 = \left(\frac{p}{\alpha_2} + \xi^2\right)^{1/2}$$

Equations (10) through (12) are now simple, ordinary differential equations whose solutions may be readily written as

$$\bar{\bar{s}}_1 = C_1 \cosh[\omega_1(z - h_1)] + \frac{Q}{2\pi l K_1 p \omega_1^2} \tag{19}$$

$$\bar{\bar{s}}_2 = C_2 \cosh[\omega_2(z + h_2)] \tag{20}$$

$$\bar{\bar{s}}_3 = A \sinh(\omega_1 z) + B \cosh(\omega_1 z) \tag{21}$$

Note that conditions (13) and (14) have already been considered in writing (19) and (20).

Constants A, B, C_1, and C_2 can be found through application of boundary conditions (15) through (18). By substituting the expressions for the above constants in (19) through (21) and performing the Hankel transform inversion, one can obtain

$$\bar{s}_1 = \frac{Q}{2\pi l K_1}\int_0^\infty \left\{\frac{1}{p\omega_1^2} - \frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \cdot [K_2\omega_2 \sinh(\omega_2 h_2)\cosh[\omega_1(h_1 - l)] + K_1\omega_1\cosh(\omega_2 h_2)\sinh[\omega_1(h_1 - l)]][FF(\omega_1, \omega_2)]^{-1}\right\} \cdot J_0(\xi r)\xi\, d\xi \tag{22}$$

$$\bar{s}_2 = \frac{Q}{2\pi l K_1} \int_0^\infty \frac{\cosh [\omega_2(z + h_2)]}{p\omega_1^2} \cdot \frac{K_1\omega_1 \sinh (\omega_1 l) J_0(\xi r)\xi \, d\xi}{FF(\omega_1, \omega_2)} \quad (23)$$

$$\bar{s}_3 = \frac{Q}{2\pi l K_1} \int_0^\infty \frac{\sinh (\omega_1 l)}{p\omega_1^2} \cdot [K_2\omega_2 \sinh (\omega_2 h_2) \sinh (\omega_1 z) + K_1\omega_1 \cosh (\omega_2 h_2) \cosh (\omega_1 z)][FF(\omega_1, \omega_2)]^{-1} \cdot J_0(\xi r)\xi \, d\xi \quad (24)$$

where

$$FF(\omega_1, \omega_2) \equiv K_2\omega_2 \sinh (\omega_2 h_2) \cosh (\omega_1 h_1) + K_1\omega_1 \cosh (\omega_2 h_2) \sinh (\omega_1 h_1)$$

Equations (22) through (24) represent the Laplace transform solutions for drawdowns in the two-layer system.

To obtain the inverse solutions of (22) through (24), let us first consider

$$\bar{G}(p) \equiv \frac{\Omega(p)}{g(p)} \equiv \frac{\cosh [\omega_1(z - h_1)]}{p\omega_1^2} \cdot [K_2\omega_2 \sinh (\omega_2 h_2) \cosh [\omega_1(h_1 - l)] + K_1\omega_1 \cosh (\omega_2 h_2) \sinh [\omega_1(h_1 - l)]] \cdot [FF(\omega_1, \omega_2)]^{-1} \quad (25)$$

If the zeros of $g(p)$ are shown by $p_1, p_2, p_3, \cdots, p_n, \cdots$ such that each of them has a different value, provided that $\Omega(p_n) \neq 0$ and $g'(p_n) \neq 0$, then the inverse transform of $\bar{G}(p)$ may be obtained from the following formula [*Jaeger*, 1949]:

$$G(t) = L^{-1}\{\bar{G}(p)\} = \sum_{n=1}^{\infty} \frac{\Omega(p_n)}{g'(p_n)} e^{p_n t} \quad (26)$$

Any of the summation terms in (26) may be replaced by

$$\left[\frac{(p - p_n)\Omega(p)}{g(p)}\right]_{p=p_n} e^{p_n t}$$

The zeros of $g(p)$, as defined in (25), are $p = 0$ and $p = -\xi^2\alpha_1$ (equivalent to $\omega_1^2 = 0$), as well as all zeros of

$$FF(\omega_1, \omega_2) = K_2\omega_2 \sinh (\omega_2 h_2) \cosh (\omega_1 h_1) + K_1\omega_1 \cosh (\omega_2 h_2) \sinh (\omega_1 h_1) = 0 \quad (27)$$

Depending on the nature of ω_1 and ω_2, four different cases must be considered.

Case 1. When both ω_1 and ω_2 are real, the left-hand side of (27) is always greater than zero, and as a result, the equation has no zeros for such a case.

Case 2. If both ω_1 and ω_2 are purely imaginary, then we may introduce the following change of variable,

$$\omega_1 = \pm i\beta/h_1 \qquad \omega_2 = \pm i\gamma/h_2$$

where β and γ are both real and positive. Equation (27) may now be written as

$$A\gamma_n \tan \gamma_n + \beta_n \tan \beta_n = 0 \quad (28)$$

where

$$A = \frac{K_2 h_1}{K_1 h_2} \qquad \beta_n = \left\{h_1^2\left[\frac{\alpha_2}{\alpha_1}\left(\frac{\gamma_n^2}{h_2^2} + \xi^2\right) - \xi^2\right]\right\}^{1/2}$$

Equation (28) has an infinite number of zeros such as γ_1, γ_2, γ_3, $\cdots$, γ_n, $\cdots$, and the corresponding values of p_n are given by

$$p_n = -\alpha_2\left(\frac{\gamma_n^2}{h_2^2} + \xi^2\right) \quad (29)$$

Note that the common roots of $\cos \gamma = 0$ and $\cos \beta = 0$, if any, are also zeros of (27).

Case 3. When ω_1 is real and ω_2 is purely imaginary, then one can set $\omega_1 = \pm\beta_n/h_1$ and $\omega_2 = \pm i\gamma_n/h_2$, where again γ_n and β_n are real and positive numbers. In this case (27) becomes

$$A\gamma_n \tan \gamma_n - \beta_n \tanh \beta_n = 0 \quad (30)$$

where

$$\beta_n = \left\{h_1^2\left[\xi^2 - \frac{\alpha_2}{\alpha_1}\left(\frac{\gamma_n^2}{h_2^2} + \xi^2\right)\right]\right\}^{1/2}$$

Equation (30) usually has a limited number of zeros.

Case 4. When ω_1 is purely imaginary and ω_2 is real, then let $\omega_1 = \pm i\beta_n/h_1$ and $\omega_2 = \pm\gamma_n/h_2$, where γ_n and β_n are both real and positive. Here (27) may be written

$$A\gamma_n \tanh \gamma_n - \beta_n \tan \beta_n = 0 \quad (31)$$

where

$$\beta_n = \left\{h_1^2\left[\frac{\alpha_2}{\alpha_1}\left(\xi^2 - \frac{\gamma_n^2}{h_2^2}\right) - \xi^2\right]\right\}^{-1/2}$$

Equation (31) also has a limited number of zeros.

Depending on the parameters of the problem, zeros of either one or two of the last three cases described above must be considered. Once the zeros are found, corresponding terms in the summation in (26) can easily be calculated. In (26) the term corresponding to $p = 0$ is

$$f(\xi) \equiv \frac{(p - 0)\Omega(0)}{g(0)} = \frac{\cosh [\xi(z - h_1)]}{\xi^2} \cdot \{K_2 \sinh (\xi h_2) \cosh [\xi(h_1 - l)] + K_1 \cosh (\xi h_2) \sinh [\xi(h_1 - l)]\} \cdot \{K_2 \sinh (\xi h_2) \cosh (\xi h_1) + K_1 \cosh (\xi h_2) \sinh (\xi h_1)\}^{-1} \quad (32)$$

and the term corresponding to $p = -\xi^2\alpha_1$ is

$$\left.\frac{(p + \xi^2\alpha_1)\Omega(p)}{g(p)} e^{-\xi^2\alpha_1 t}\right|_{p=-\xi^2\alpha_1} = -\frac{1}{\xi^2} e^{-\xi^2\alpha_1 t} \quad (33)$$

Therefore (26) may be written as

$$G(t) = f(\xi) - \frac{1}{\xi^2} e^{-\xi^2\alpha_1 t} + \sum_{n=1}^{\infty} \frac{\Omega(p_n)}{g'(p_n)} e^{p_n t} \quad (34)$$

where p_n are now only roots of (27).

Noting that

$$L^{-1}\left\{\frac{1}{p\omega_1^2}\right\} = \frac{1}{\xi^2}(1 - e^{-\alpha_1\xi^2 t}) \quad (35)$$

the inverse Laplace transform of (22) may now be written

$$s_1(t) = L^{-1}\{\bar{s}_1(p)\}$$

$$= \frac{Q}{2\pi l K_1} \int_0^\infty J_0(\xi r)\xi \left(\frac{1}{\xi^2} - f(\xi) - \sum_{n=1}^\infty \frac{\Omega(p_n)}{g'(p_n)} e^{p_n t} \right) d\xi \qquad (36)$$

By introducing the following dimensionless parameters:

$$S_D = 4\pi K_1 h_1 s/Q \qquad t_D = \alpha_1 t/r^2 \qquad r_D = r/h_1$$

$$z_D = z/h_1 \qquad l_D = l/h_1 \qquad H = h_2/h_1 \qquad D = \alpha_2/\alpha_1$$

$$A = K_2 h_1/K_1 h_2 \qquad x = \xi h_1$$

(36) becomes

$$s_{D_1} = \frac{2}{l_D} \int_0^\infty x J_0(x r_D) \left[\frac{1}{x^2} - f_1(x) + \sum_{n=1}^\infty \frac{B_1'}{A'} \cdot \exp\left\{ -\left(\frac{\gamma_n^2}{H^2} + x^2 \right) D r_D^2 t_D \right\} \right] dx \qquad (37)$$

where

$$f_1(x) = \frac{\cosh[x(1 - z_D)]}{x^2} \cdot \{AH \tanh(Hx) \cosh[x(1 - l_D)] + \sinh[x(1 - l_D)]\}\{AH \tanh(Hx) \cosh(x) + \sinh(x)\}^{-1} \qquad (38)$$

and the expressions for A' and B_1' depend on the nature of ω_1 and ω_2. If both ω_1 and ω_2 are pure imaginary, then

$$A' = \left(\frac{1}{2}\right)\left(\frac{\gamma_n^2}{H^2} + x^2\right)\left\{\beta_n \left[D - H^2 \left(\frac{\beta_n}{\gamma_n}\right)^2 \right] \cos\gamma_n \sin\beta_n + \beta_n^2 (AH^2 + D) \cos\gamma_n \cos\beta_n - \left(H^2 \frac{\beta_n}{\gamma_n} + AD \frac{\gamma_n}{\beta_n} \right) \beta_n^2 \sin\gamma_n \sin\beta_n \right\} \qquad (39)$$

$$B_1' = \cos[\beta_n(z_D - 1)]\{A\gamma_n \sin\gamma_n \cos[\beta_n(1 - l_D)] + \beta_n \cos\gamma_n \sin[\beta_n(1 - l_D)]\} \qquad (40)$$

If either ω_1 or ω_2 becomes real, then β_n or γ_n in (39) and (40) should be replaced by $(i\beta_n)$ or $(i\gamma_n)$, respectively.

One can find the inversion of $\bar{s}_2$ and $\bar{s}_3$ in a similar manner. In dimensionless forms, the solutions become

$$s_{D_2} = \frac{2}{l_D} \int_0^\infty x J_0(x r_D) \left\{ f_2(x) + \sum_{n=1}^\infty \frac{B_2'}{A'} \cdot \exp\left[-\left(\frac{\gamma_n^2}{H^2} + x^2 \right) D r_D^2 t_D \right] \right\} dx \qquad (41)$$

$$s_{D_3} = \frac{2}{l_D} \int_0^\infty x J_0(x r_D) \left\{ f_3(x) + \sum_{n=1}^\infty \frac{B_3'}{A'} \cdot \exp\left[-\left(\frac{\gamma_n^2}{H^2} + x^2 \right) D r_D^2 t_D \right] \right\} dx \qquad (42)$$

where

$$f_2(x) = \frac{1}{x^2} \cdot \{\cosh[x(z_D + H)] \sinh(x l_D)\}\{AH \sinh(xH) \cdot \cosh(x) + \cosh(xH) \sinh(x)\}^{-1} \qquad (43)$$

$$f_3(x) = \frac{\sinh(x l_D)}{x^2} \cdot \{AH \sinh(xH) \sinh(x z_D) + \cosh(xH) \cdot \cosh(x z_D)\}\{AH \sinh(xH) \cosh(x) + \cosh(xH) \sinh(x)\}^{-1} \qquad (44)$$

and when ω_1 and ω_2 are both pure imaginary,

$$B_2' = -\beta_n \sin(\beta_n l_D) \cos\left[\gamma_n \left(1 + \frac{z_D}{H} \right) \right] \qquad (45)$$

$$B_3' = \sin(\beta_n l_D)\{A\gamma_n \sin\gamma_n \sin(\beta_n z_D) - \beta_n \cos\gamma_n \cos(\beta_n z_D)\} \qquad (46)$$

Here, too, if either ω_1 of ω_2 becomes real, then β_n or γ_n in (45) and (46) should be replaced by $(i\beta_n)$ or $(i\gamma_n)$, respectively.

Solution for single-layer case. The solution for a single-layer aquifer with a partially penetrating well can be obtained from the two-layer solution by letting the hydraulic conductivity of the lower layer vanish. Letting $K_2 = 0$, (22) becomes

$$\bar{s}_1 = \frac{Q}{2\pi l K_1} \int_0^\infty \left\{ \frac{1}{p\omega_1^2} - \frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \cdot \frac{\sinh[\omega_1(h_1 - l)]}{\sinh(\omega_1 h_1)} \right\} \xi J_0(\xi r)\, d\xi \qquad (47)$$

$\bar{s}_3$ in (24) can also be written in a form identical with (47) if one sets $K_2 = 0$.

The Appendix shows that the inverse of $\bar{s}_1$ may be written as

$$s_1 = \frac{Q}{4\pi K_1 h_1} \left[\int_u^\infty \frac{e^{-y}}{y} dy + \frac{2h_1}{\pi l} \sum_{n=1}^\infty \frac{1}{n} \sin\frac{n\pi l}{h_1} \cdot \cos\frac{n\pi(z - h_1)}{h_1} \int_u^\infty \exp\left\{ -y - \frac{(rn\pi)^2}{4yh_1^2} \right\} \frac{dy}{y} \right] \qquad (48)$$

Equation (48) is exactly the same as that given by *Hantush* [1957].

Solution for small values of time. To find a solution for the early stages of pumping, one has to look for sufficiently large values of p corresponding to small values of t. Let us consider the second part of the integrand in (22). By rearranging this term, one gets

$$\frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \cdot \{K_2\omega_2 \sinh(\omega_2 h_2) \cosh[\omega_1(h_1 - l)] + K_1\omega_1 \cosh(\omega_2 h_2 \sinh[\omega_1(h_1 - l)]\} \cdot \{K_2\omega_2 \sinh(\omega_2 h_2) \cosh(\omega_1 h_1) + K_1\omega_1 \cosh(\omega_2 h_2) \sinh(\omega_1 h_1)\}^{-1}$$

$$= \frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \frac{\sinh[\omega_1(h_1 - l)]}{\sinh(\omega_1 h_1)}$$

$$\cdot \{K_2\omega_2 \tanh(\omega_2 h_2) \coth[\omega_1(h_1 - l)] + K_1\omega_1\}$$

$$\cdot \{K_2\omega_2 \tanh(\omega_2 h_2) \coth(\omega_1 h_1) + K_1\omega_1\}^{-1} \quad (49)$$

Noting that coth (x) is almost equal to unity for all values of $x > 3$, the right-hand side of (49) may be simplified to

$$\frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \frac{\sinh[\omega_1(h_1 - l)]}{\sinh(\omega_1 h_1)}$$

provided that $\omega_1(h_1 - l) \geq (10)^{1/2}$. As a result, under this condition, (22) becomes

$$\bar{s}_1 = \frac{Q}{2\pi l K_1} \int_0^\infty \left\{\frac{1}{p\omega_1^2} - \frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \cdot \frac{\sinh[\omega_1(h_1 - l)]}{\sinh(\omega_1 h_1)}\right\} \xi J_0(\xi r)\, d\xi$$

which is identical with (47) and therefore leads to the solution for the single-layer partial penetration problem.

The above condition may be expressed in terms of real time. Recalling the definition of ω_1, we can write

$$(h_1 - l)^2\left(\frac{p}{\alpha_1} + \xi^2\right) \geq \frac{p}{\alpha_1}(h_1 - l)^2 \geq 10$$

or

$$t \leq \frac{(h_1 - l)^2}{10\alpha_1}$$

In terms of dimensionless parameters, this becomes

$$t_D \leq \frac{(1 - l_D)^2}{10 r_D^2} \quad (50)$$

For values of t_D that are less than this, the two-layer aquifer behaves as if the lower layer were absent, or in other words, the transient effects have not yet reached the interface.

Solution for large values of time. To obtain a solution for large values of time, we shall examine the case when p is small. One may note that at large values of time and provided that $r \geq 1.5[h_1 + K_2h_2/K_1]$, only small values of ξ make a major contribution. Since $\sinh x \approx x$ and $\cosh x \approx 1$ when $x < 0.01$, (22) may be simplified to the following for sufficiently large values of time, provided that $r \geq 1.5[h_1 + K_2h_2/K_1]$:

$$\bar{s}_1 = \frac{Q}{2\pi l K_1} \int_0^\infty \left[\frac{1}{p\omega_1^2} - \frac{1}{p\omega_1^2} \frac{h_2K_2\omega_2^2 + K_1\omega_1^2(h_1 - l)}{h_2K_2\omega_2^2 + K_1\omega_1^2 h_1}\right] \cdot J_0(\xi r)\xi\, d\xi \quad (51)$$

After simplification, (51) becomes

$$\bar{s}_1 = \frac{Q}{2\pi K_1 h_1} \int_0^\infty \frac{1}{p}(J_0(\xi r)\xi\, d\xi) \cdot \left[\left(\frac{h_2K_2}{h_1K_1}\right)\left(\frac{p}{\alpha_2} + \xi^2\right) + \left(\frac{p}{\alpha_1} + \xi^2\right)\right]^{-1} \quad (52)$$

From the tables of Laplace transforms, one can easily find that

$$s_1 = \frac{Q}{2\pi(K_1h_1 + K_2h_2)} \int_0^\infty J_0(\xi r)\xi \left\{\left[1 - \exp\left(-\frac{h_1K_1 + h_2K_2}{S_{s_1}h_1 + S_{s_2}h_2}\xi^2 t\right)\right][\xi^2]^{-1}\right\} d\xi \quad (53)$$

Equation (53) may be written as [*Javandel*, 1982]

$$s_1 = \frac{Q}{4\pi(K_1h_1 + K_2h_2)} \int_v^\infty \frac{e^{-y}}{y}\, dy \quad (54)$$

where

$$v = \frac{r^2(S_{s_1}h_1 + S_{s_2}h_2)}{4t(T_1 + T_2)}$$

Note that (54) has the form of the well-known exponential integral. In dimensionless form, we then have

$$s_{D_1} = \frac{1}{1 + (T_2/T_1)} \int_v^\infty \frac{e^{-y}}{y}\, dy \quad (55)$$

Since we are dealing with large values of time, (55) may be approximated by

$$s_{D_1} \approx \frac{2.3}{1 + (T_2/T_1)}\left(\log t_D + \log \frac{2.25(1 + T_2/T_1)}{1 + (S_{s_2}h_2/S_{s_1}h_1)}\right) \quad (56)$$

This is a very interesting result because it indicates that a semilog plot of dimensionless drawdown versus dimensionless time will yield a straight line when the pumping time becomes sufficiently large. The slope of this line will be

$$m = \frac{2.3}{1 + T_2/T_1} \quad (57)$$

and when $r \geq 1.5[h_1 + (K_2h_2)/K_1]$, the value of t_D corresponding to $s_{D_1} = 0$ will be given by

$$t_{D_0} = \frac{1 + (S_{s_2}h_2/S_{s_1}h_1)}{2.25(1 + T_2/T_1)} \quad (58)$$

Although (58) holds for $r \geq 1.5[h_1 + (K_2h_2)/K_1]$, (57) is true for all values of r. Note that the expression $r \geq 1.5[h_1 + (K_2h_2)/K_1]$ is only approximate and is generally conservative. For large values of time and at radial distances beyond $r \geq 1.5[h_1 + (K_2h_2)/K_1]$, the equipotentials in the aquifer are almost vertical. This means the effect of partial penetration beyond that radial distance could be ignored, particularly if the drawdown is measured in an observation well rather than in a piezometer.

Another important result that can be drawn from (54) is that if we introduce a new set of dimensionless definitions for drawdown and time in the following form:

$$\tilde{s}_{D_1} = \frac{4\pi(T_1 + T_2)}{Q} s_1 \quad (59)$$

and

$$\tilde{t}_D = \frac{t(T_1 + T_2)}{r^2(S_{s_1}h_1 + S_{s_2}h_2)} \quad (60)$$

then plots of $\tilde{s}_{D_1}$ versus $\tilde{t}_D$ for two-layer aquifers at large values of time and $r \geq 1.5[h_1 + (K_2h_2)/K_1]$ will coincide with the Theis curve.

WELL OPEN IN THE LOWER LAYER

When the pumping well is open along length l in the upper part of the lower layer, as illustrated in part *a* of Figure 2, a set of solutions for s_1, s_2, and s_3 can be obtained using the same approach discussed above. By symmetry, these solutions will also apply to the situation illustrated in part *b* of Figure 2.

In the Laplace domain the solutions for the first case in each of the three layers are as given below:

$$\bar{s}_1 = \frac{Q}{2\pi l K_2}\int_0^\infty \left\{\frac{\cosh[\omega_1(z-h_1)]}{p\omega_2^2} \cdot \frac{K_2\omega_2\{\sinh(\omega_2h_2) - \sinh[\omega_2(h_2-l)]\}}{FF(\omega_1,\omega_2)}\right\} J_0(\xi r)\xi\, d\xi \qquad (61)$$

$$\bar{s}_2 = \frac{Q}{2\pi l K_2}\int_0^\infty \frac{J_0(\xi r)\xi}{p\omega_2^2}\{1 - [1/FF(\omega_1,\omega_2)]\{K_1\omega_1 \sinh(\omega_1h_1) \cdot \cosh[\omega_2(z+h_2)] + \sinh[\omega_2(h_2-l)]\,[K_2\omega_2\cosh(\omega_1h_1) \cdot \cosh(\omega_2 z) - K_1\omega_1\sinh(\omega_2 z)\sinh(\omega_1h_1)]\}\}\, d\xi \qquad (62)$$

$$\bar{s}_3 = \frac{Q}{2\pi l K_2}\int_0^\infty \frac{\cosh[\omega_2(z+h_2)]}{p\omega_2^2}[-K_1\omega_1\sinh(\omega_1h_1) + K_1\omega_1\sinh(\omega_1h_1)\cosh(\omega_2 l) + K_2\omega_2\cosh(\omega_1h_1) \cdot \sinh(\omega_2 l)][FF(\omega_1,\omega_2)]^{-1} J_0(\xi r)\xi\, d\xi \qquad (63)$$

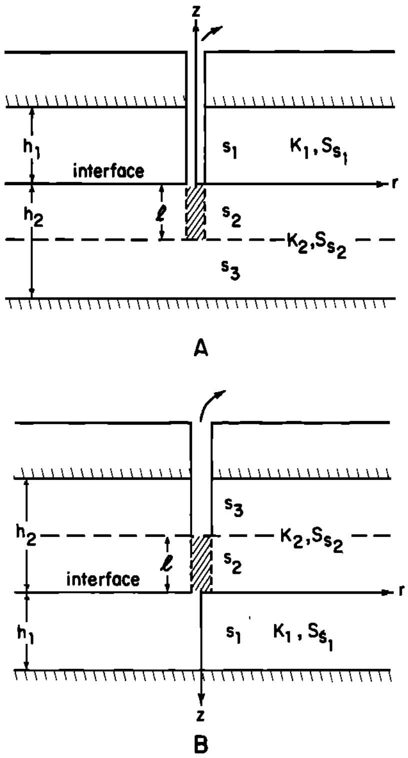

Fig. 2. Schematic diagram of two-layer aquifers, (*a*) with a partially penetrating well open only in the top of the lower layer and (*b*) when the pumping well is open in the bottom of the top layer.

It can be easily verified that (61)–(63) satisfy the appropriate boundary conditions. The Laplace inversion of the above expressions can be readily obtained through the same procedure discussed above. With regard to the inversion of $\bar{s}_1$, note that the nonremovable zeros of the denominator in the integrand are $p = 0$ as well as all the roots of $FF(\omega_1, \omega_2) = 0$. Finally, the solutions for s_1, s_2, and s_3 in nondimensional form are

$$s_{D_1} = \frac{2}{\hat{l}_D}\int_0^\infty \hat{\xi} J_0(\hat{\xi}\hat{r}_D) \cdot \left\{R_1(\hat{\xi}) + \sum_{n=1}^\infty \frac{q_1}{\delta}\exp[-(\gamma_n^2 + \hat{\xi}^2)\hat{t}_D\hat{r}_D^2]\right\} d\hat{\xi} \qquad (64)$$

$$s_{D_2} = \frac{2}{\hat{l}_D}\int_0^\infty \hat{\xi} J_0(\hat{\xi}\hat{r}_D) \cdot \left\{\frac{1}{\hat{\xi}^2} - R_2(\hat{\xi}) + \sum_{n=1}^\infty \frac{q_2}{\delta}\exp[-(\gamma_n^2 + \hat{\xi}^2)\hat{t}_D\hat{r}_D^2]\right\} d\hat{\xi} \qquad (65)$$

$$s_{D_3} = \frac{2}{\hat{l}_D}\int_0^\infty \hat{\xi} J_0(\hat{\xi}\hat{r}_D) \cdot \left\{R_3(\hat{\xi}) + \sum_{n=1}^\infty \frac{q_3}{\delta}\exp[-(\gamma_n^2 + \hat{\xi}^2)\hat{t}_D\hat{r}_D^2]\right\} d\hat{\xi} \qquad (66)$$

where

$$R_1(\hat{\xi}) = \frac{\cosh[\hat{\xi}(\hat{z}_D - \hat{H})]}{\hat{\xi}^2}\{\sinh(\hat{\xi}) - \sinh[\hat{\xi}(1-\hat{l}_D)]\} \cdot \{\cosh(\hat{\xi}\hat{H})\sinh(\hat{\xi}) + (\hat{H}/A)\sinh(\hat{\xi}\hat{H})\cosh(\hat{\xi})\}^{-1} \qquad (67)$$

$$R_2(\hat{\xi}) = \{\hat{\xi}^2[\cosh(\hat{\xi}\hat{H})\sinh(\hat{\xi}) + (\hat{H}/A)\sinh(\hat{\xi}\hat{H})\cosh(\hat{\xi})\}^{-1} \cdot \{(\hat{H}/A)\sinh(\hat{\xi}\hat{H})\cosh[\hat{\xi}(\hat{z}_D+1)] + \sinh[\hat{\xi}(1-\hat{l}_D)] \cdot [\cosh(\hat{\xi}\hat{H})\cosh(\hat{\xi}\hat{z}_D) - (\hat{H}/A)\sinh(\hat{\xi}\hat{z}_D)\sinh(\hat{\xi}\hat{H})]\} \qquad (68)$$

$$R_3(\hat{\xi}) = \frac{\cosh[\hat{\xi}(\hat{z}_D+1)]}{\hat{\xi}^2}[-(\hat{H}/A)\sinh(\hat{\xi}\hat{H}) + (\hat{H}/A)\sinh(\hat{\xi}\hat{H})\cosh(\hat{\xi}\hat{l}_D) + \cosh(\hat{\xi}\hat{H})\sinh(\hat{\xi}\hat{l}_D)] \cdot [\cosh(\hat{\xi}\hat{H})\sinh(\hat{\xi}) + (\hat{H}/A)\sinh(\hat{\xi}\hat{H})\cosh(\hat{\xi})]^{-1} \qquad (69)$$

If ω_1 and ω_2 are both pure imaginary,

$$q_1 = -\gamma_n \cos\left[\beta_n\left(\frac{\hat{z}_D}{\hat{H}} - 1\right)\right]\{\sin\gamma_n - \sin[\gamma_n(1-\hat{l}_D)]\} \qquad (70)$$

$$q_2 = \sin[\gamma_n(1-\hat{l}_D)]\{\gamma_n\cos\beta_n\cos(\gamma_n\hat{z}_D) + (\beta_n/A)\sin(\gamma_n\hat{z}_D)\sin\beta_n\} + (\beta_n/A)\sin\beta_n\cos[\gamma_n(\hat{z}_D+1)] \qquad (71)$$

$$q_3 = \cos[\gamma_n(\hat{z}_D+1)]\{(\beta_n/A)\sin\beta_n - (\beta_n/A)\sin\beta_n\cos(\gamma_n\hat{l}_D) - \gamma_n\cos\beta_n\sin(\gamma_n\hat{l}_D)\} \qquad (72)$$

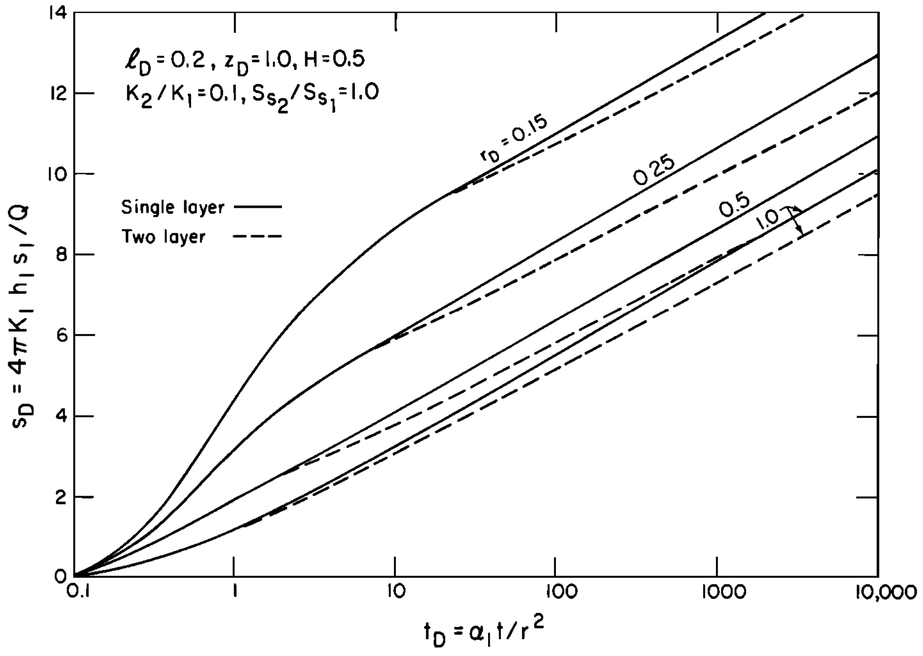

Fig. 3. Dimensionless drawdown versus dimensionless time for a two-layer aquifer compared with a single-layer system.

$$\delta = \frac{1}{2}(\gamma_n^2 + \hat{k}^2)\left\{\beta_n \sin\beta_n \cos\gamma_n\left(\frac{D\hat{H}^2\gamma_n^2}{A\beta_n^2} - \frac{1}{A}\right)\right.$$

$$- \gamma_n^2 \sin\beta_n \sin\gamma_n\left(\frac{D\hat{H}^2\gamma_n}{\beta_n} + \frac{\beta_n}{A\gamma_n}\right)$$

$$\left. + \gamma_n^2 \cos\beta_n \cos\gamma_n\left(1 + \frac{D\hat{H}^2}{A}\right)\right\} \qquad (73)$$

If either ω_1 or ω_2 is real, then β_n or γ_n in (70)–(73) should be replaced by $(i\beta_n)$ or $(i\gamma_n)$, respectively.

Examination of (62) reveals that if we let the hydraulic conductivity of the top layer vanish, the solution for s_2 converges to the case for a single layer with partial penetration. However, in contrast to the previous case where the well was open only in the upper part of the top layer, in this case, due to the direct contact between the two layers, the solution for s_2 at small values of time cannot in general be closely approximated by a single-layer solution. Only when the ratio of K_1/K_2 becomes very small does the single-layer solution apply to the lower layer.

DISCUSSION OF RESULTS

Figures 3 through 5 show semilog plots of dimensionless drawdown versus dimensionless time for several selected parameters. Solutions for single-layer partial penetration [*Witherspoon et al.*, 1967] have also been included for the sake of comparison. One notes that at early values of time the solutions for the two-layer aquifer coincides with that of the single-layer case. This was shown to be the case from the general solution. At large values of time, the slopes of the curves are in agreement with the values obtained from (57). As is apparent on these figures, the slopes of the curves must converge to $m = 2.3$ when $T_2 = 0$, which of course corresponds to the single-layer case for large values of time.

Figure 6 was prepared to check the validity of (58). Dimensionless drawdown versus dimensionless time was computed for the special case where $l_D = 0.5$, $K_2/K_1 = 0.5$, $h_2/h_1 = 0.5$, $S_{s_1} = S_{s_2}$, and $z_D = 1.0$ for values of $r_D = 0.2$, 0.3, 1.0, and 1.5. As discussed above, when $r \geq 1.5[h_1 + (K_2h_2)/K_1]$, at large values of time the system behaves essentially like a single-layer, homogeneous aquifer such that $T_{eq} = T_1 + T_2$ and $S_{eq} = S_{s_1}h_1 + S_{s_2}h_2$. If we substitute the appropriate values into (58), we obtain

$$T_{D_0} = \frac{1 + S_{s_2}h_2/S_{s_1}h_1}{2.25(1 + T_2/T_1)} = \frac{1 + 0.5}{2.25(1 + 0.25)} = 0.53$$

To check this, the straight line for $r_D = 1.5$ was extrapolated to $s_D = 0$, and as indicated on Figure 6, the intercept gives 0.53. Although the real value of r in this case is somewhat smaller than $1.5[h_1 + (K_2h_2)/K_1]$, nevertheless, this example shows that (58) still holds and indicates that the expression $r \geq 1.5[h_1 + (K_2h_2)/K_1]$ is on the conservative side.

A finite element model was also used to provide a numerical approach to this same two-layer problem. Figure 7 shows a comparison of dimensionless results for $r_D = 0.1$, 0.2, and 1.0. Two different meshes were used to duplicate the conditions of the analytical model. The first mesh only included 323 nodal points, and it is evident that computed drawdowns at any given time were too low for $r_D = 0.1$ and 0.2 and too high for $r_D = 1.0$. A second mesh using 681 nodal points gave much better agreement with the analytical results. This suggests that some care must be exercised when approaching this kind of complex problem from the numerical standpoint.

APPLICATION TO AQUIFER PUMP TESTS

If a well is completed through the total thickness of a two-layer aquifer and is pumped at constant rate, the analysis of the results can only yield the transmissivity and storativity of the equivalent system. However, if the well is completed in only one part of either layer of the system, the following

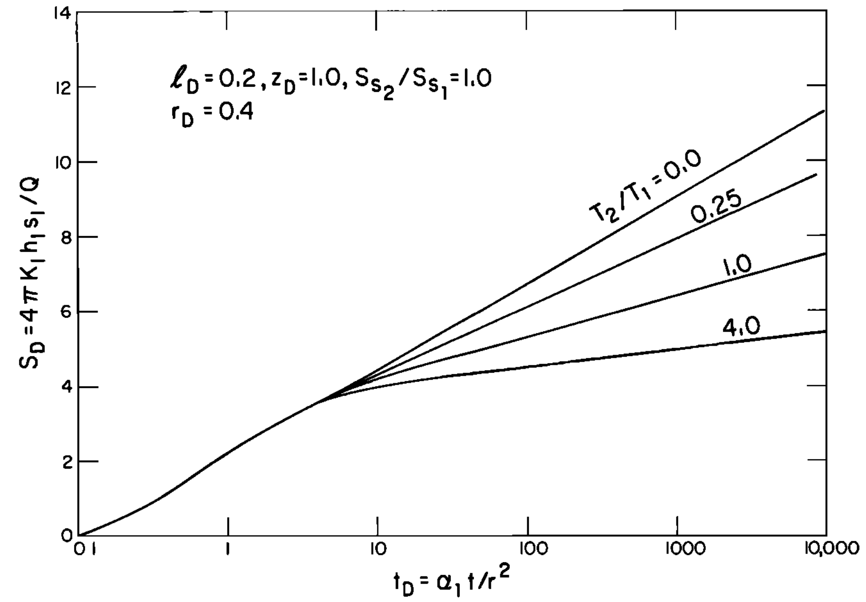

Fig. 4. Effect of transmissivity contrast on drawdowns of a two-layer system.

procedure can be used to investigate the hydrological properties of the individual layers. The properties of the layer being pumped can be determined from the early time response at an appropriately located observation well. Since late time response reflects the properties of the combined system, one can then determine the properties of the unpumped layer.

Properties of pumped layer. Two different methods can be used to obtain results for the pumped layer: (1) the inflection method and (2) the type curve method. We shall present both below and then discuss a method of determining the properties of the unpumped layer.

Inflection method. This method has been introduced by *Hantush* [1961*a*] for a single-layer aquifer with partial penetration and will be reviewed briefly. One should construct a semilog plot of drawdown data from the piezometer versus time. If an inflection point is clearly indicated, a tangent to the curve at the point of inflection can be used to determine the slope of the curve $m_{in}' = \Delta s$/cycle. Hantush has shown that

$$\beta^2/\sqrt{\pi} = xe^{x^2} \operatorname{erf}(x) \tag{74}$$

When the piezometer is open at the top of the aquifer, or when $r > l$ and the depth to the opening in the piezometer is less than that of the pumping well, then $\beta = l/r$. On the other hand, when the piezometer is open at approximately the elevation of the bottom of the pumping well, $\beta = 2l/r$. Knowing β, one can evaluate x from (74) and then compute

$$u_i = (x/\beta)^2 \tag{75}$$

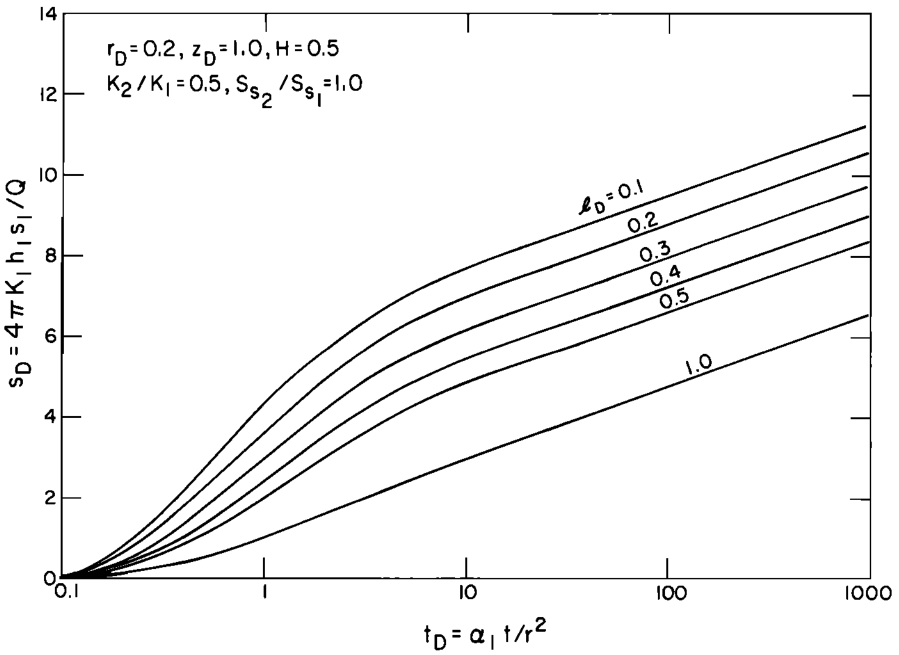

Fig. 5. Effect of penetration length of the pumping well on drawdowns of a two-layer system.

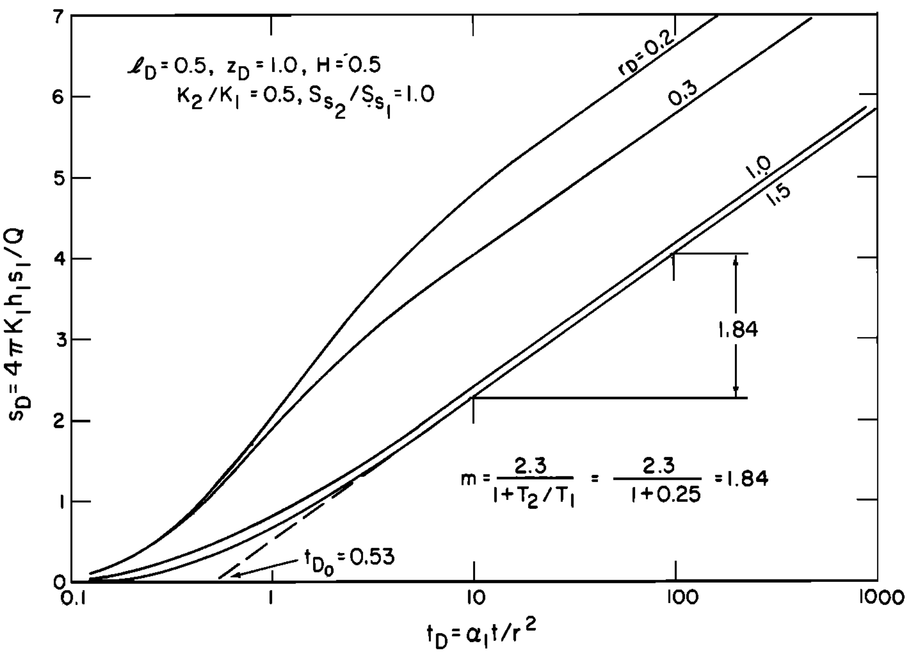

Fig. 6. Dimensionless drawdown versus dimensionless time for a two-layer system illustrating the slope and intercept time of a curve.

The hydraulic conductivity K_1 can then be determined from

$$K_1 = \frac{2.3\,Q}{4\pi m_{in}'l} e^{-u_i} \operatorname{erf}(x) \qquad \beta = l/r$$

or

$$K_1 = \frac{2.3\,Q}{8\pi m_{in}'l} e^{-u_i} \operatorname{erf}(x) \qquad \beta = 2l/r$$

The next step is to evaluate a function that *Hantush* [1961*b*] defined as

$$M(u_i, \beta) = \int_{u_i}^{\infty} \frac{e^{-y}}{y} \operatorname{erf}(\beta\sqrt{y})\, dy \qquad (76)$$

and that has been tabulated [*Hantush*, 1961*b*; *Witherspoon et al.*, 1967]. One then calculates drawdown at the inflection

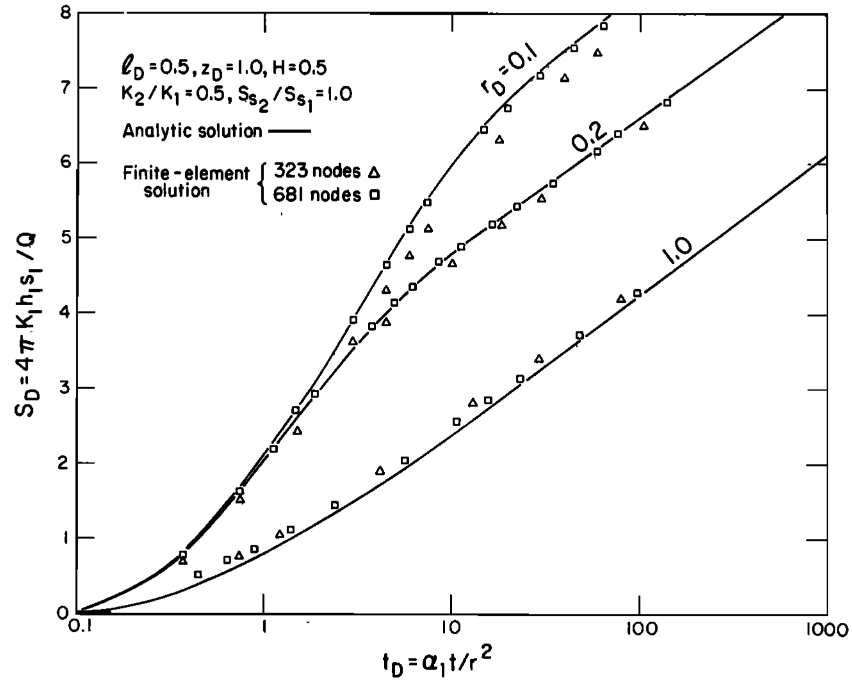

Fig. 7. Comparison of the analytic solution with finite-element calculations.

point from

$$s_{in} = \frac{Q}{B\pi K_1 l} M(u_i, \beta) \quad (77)$$

where $B = 4$ when $\beta = l/r$ and $B = 8$ when $\beta = 2l/r$.

From the semilog plot of drawdown data, one reads the time t_{in} corresponding to the value of s_{in}. Finally, the specific storage can be computed from

$$S_{s_1} = 4K_1 t_{in} u_i / r^2 \quad (78)$$

Type curve method. This method is essentially the same as the standard log-log type curve method except that one must prepare a special type curve of s_D versus t_D for the appropriate parameters of the system. The method will again yield K_1 and S_{s_1}.

The application of the inflection and type curve methods are based on the assumption that the early time response of the two-layer system is essentially controlled by the properties of the pumped layer. This is generally the case when the piezometer is located at a radial distance from the pumping well that is less than half the thickness of the pumped layer.

Properties of unpumped layer. To obtain the hydraulic properties of the unpumped layer, the semilog plot of observed drawdown versus time should reveal a straight line which reflects the properties of the combined system if the pumping test has been run for a sufficiently long period of time. The final slope m' of this straight line can be used to obtain $(K_1h_1 + K_2h_2)$ from

$$m' = \frac{2.3Q}{2\pi(K_1h_1 + K_2h_2)} \quad (79)$$

Since K_1 has been evaluated and h_1 is known, (K_2h_2) is readily calculated. If h_2 is also known, K_2 is easily determined.

If one needs the specific storage for the umpumped layer, it will probably be necessary to have an observation well at some distance from the pumping well such that $r \geq 1.5[h_1 + (K_2h_2)/K_1]$. As illustrated in Figure 6, an extrapolation of the straight line portion of the semilog plot back to the axis for zero drawdown can be used to determine the intercept time t_0. Then from (58), one can derive

$$S_{s_2}h_2 = \frac{2.25\, t_0(T_1 + T_2)}{r^2} - S_{s_1}h_1 \quad (80)$$

S_{s_1} can be determined from either inflection method or the usual type curve analysis of early time results, and h_1 is presumably known. Thus $(S_{s_2}h_2)$ is easily obtained from (80), and if h_2 is known, one has S_{s_2}.

In the event there is only one observation well whose $r \geq 1.5[h_1 + (K_2h_2)/K_1]$, the early drawdown data may not be sufficient to determine the properties of the pumped layer. In this case, one can only determine the hydraulic properties of the equivalent system.

CONCLUSIONS

An analytic solution to the problem of transient flow toward a partially penetrating well in a two-layer aquifer has been presented. Solutions have been developed for a pumping well that is open in either layer. The solutions have been evaluated numerically and graphical results for some typical cases are presented. The results have also been checked by comparison with finite element calculations. It was shown that the solutions reduce to the case for a single layer with partial penetration. Asymptotic solutions for small and large values of time have been developed to show that (1) at early times with partial penetration the behavior of the pumped layer is exactly the same as that of a single layer and (2) at large values of time, a semilog plot of drawdown versus time yields a straight line whose slope is only a function of the ratio T_2/T_1. Finally, a method of analyzing field data to determine the hydraulic properties of both the pumped and unpumped layers is proposed.

APPENDIX: SOLUTION FOR SINGLE-LAYER CASE

Here we show a procedure which would lead to the inverse Laplace transform of (47). The second part of the integrand in (47) may be written as

$$\lambda \equiv \frac{\cosh[\omega_1(z - h_1)]}{\omega_1^2}\frac{\sinh[\omega_1(h_1 - l)]}{\sinh(\omega_1 h_1)}$$

$$= \frac{\sinh[\omega_1(z - l)] + \sinh[\omega_1(2h_1 - l - z)]}{2\omega_1^2 \sinh(\omega_1 h_1)}$$

Now, by using the following formula from a Laplace transform table,

$$L^{-1}\left\{\frac{\sinh x\sqrt{p}}{p \sinh a\sqrt{p}}\right\} = \frac{x}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\exp\left(-\frac{n^2\pi^2 t}{a^2}\right)\sin\frac{n\pi x}{a}$$

the inverse Laplace transform of λ may be written as

$$L^{-1}\{\lambda\} = \alpha_1 e^{-\alpha_1\xi^2 t}\left\{\frac{h_1 - l}{h_1} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\exp - \frac{n^2\pi^2\alpha_1 t}{h_1^2} \cdot \sin\frac{n\pi(h_1 - l)}{h_1}\cos\frac{n\pi(z - h_1)}{h_1}\right\}$$

Since

$$L^{-1}\left(\frac{1}{\omega_1^2}\right) = \alpha_1 \exp\{-\alpha_1\xi^2 t\}$$

then

$$L^{-1}\left\{\frac{1}{\omega_1^2} - \lambda\right\}$$

$$= \alpha_1 \exp(-\alpha_1\xi^2 t)\left\{\frac{l}{h_1} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\exp\left(-\frac{n^2\pi^2\alpha_1 t}{h_1^2}\right) \cdot \sin\frac{n\pi(h_1 - l)}{h_1}\cos\frac{n\pi(z - h_1)}{h_1}\right\}$$

Noting that

$$\sin\frac{n\pi(h_1 - l)}{h_1} = -(-1)^n \sin\frac{n\pi l}{h_1}$$

we may write

$$s_1 = \frac{Q}{2\pi l K_1} \int_0^\infty J_0(\xi r)\xi \int_0^t \left\{ \frac{l\alpha_1}{h_1} \exp(-\alpha_1 \xi^2 \tau) + \frac{2\alpha_1}{\pi} \sum_{n=1}^\infty \frac{1}{n} \sin \frac{n\pi l}{h_1} \cos \frac{n\pi(z-h_1)}{h_1} \cdot \exp(-\alpha_1 \xi^2 \tau) \exp\left(-\frac{n^2\pi^2\alpha_1\tau}{h_1^2} \right) \right\} d\tau \, d\xi$$

By changing the order of integration, performing integration with respect to ξ, and applying the following formula

$$\int_0^\infty x e^{-ax^2} J_0(bx)\, dx = \frac{1}{2a} \exp\left(-\frac{b^2}{4a} \right)$$

we obtain

$$s_1 = \frac{Q}{2\pi l K_1} \int_0^t \left\{ \frac{l}{2h_1\tau} \exp\left(-\frac{r^2}{4\alpha_1\tau} \right) + \frac{2}{\pi} \sum_{n=1}^\infty \frac{1}{n} \sin \frac{n\pi l}{h_1} \cdot \cos \frac{n\pi(z-h_1)}{h_1} \cdot \frac{1}{2\tau} \exp\left(-\frac{r^2}{4\alpha_1\tau} - \frac{n^2\pi^2\alpha_1\tau}{h_1^2} \right) \right\} d\tau$$

By introducing the following change of variables

$$y = \frac{r^2}{4\alpha_1\tau} \qquad u = \frac{r^2}{4\alpha_1 \tau}$$

finally, the Laplace inversion of (47) may be written

$$s_1 = \frac{Q}{4\pi K_1 h_1} \left\{ \int_0^\infty \exp(-y) \frac{dy}{y} + \frac{2h_1}{\pi l} \sum_{n=1}^\infty \frac{1}{n} \sin \frac{n\pi l}{h_1} \cdot \cos \frac{n\pi(z-h_1)}{h_1} \int_u^\infty \exp\left(-y - \frac{(rn\pi)^2}{4yh_1^2} \right) \frac{dy}{y} \right\}$$

NOTATION

$A = K_2h_1/K_1h_2$.
$D = \alpha_2/\alpha_1$.
h_1 thickness of the top layer, m.
h_2 thickness of the lower layer, m.
$H = h_2/h_1$.
$\hat{H} = h_1/h_2$.
$J_0(x)$ Bessel's function of the first kind and zero order.
K_1, K_2 hydraulic conductivity of upper and lower layers, respectively, m/s.
l depth of penetration, m.
$l_D = l/h_1$.
$\hat{l}_D = l/h_2$.
L^{-1} Laplace transform inversion operator.
m final slope of the dimensionless time–drawdown curve.
m' final slope of the time-drawdown curve of the observed data, m/cycle.
m_{in}' slope of the tangent at the inflection point of time-drawdown curve of the observed data, m/cycle.
p Laplace transform parameter, s^{-1}.
Q rate of discharge, m^3/s.
r radial distance, m.
$r_d = r/h_1$.
$\hat{r}_D = r/h_2$.
S_{s_1}, S_{s_2} specific storage of the upper and lower layer, respectively, m^{-1}.
S_{eq} equivalent storage coefficient.
s_i drawdown of different layers, m.
s_{D_i} dimensionless drawdown, equal to $4\pi K_1h_1s_i/Q$.
$\hat{s}_{D_i}$ dimensionless drawdown, equal to $4\pi K_2h_2s_i/Q$.
$\bar{s}_{D_1} = 4\pi(T_1 + T_2)s_1/Q$.
$\bar{s}_i$ Laplace transform of s_i.
$\bar{\bar{s}}_i$ Hankel transform of $\bar{s}_i$.
s_{in} drawdown at the inflection point, m.
t time, s.
t_D dimensionless time, equal to $\alpha_1 t/r^2$.
$\hat{t}_D$ dimensionless time, equal to $\alpha_2 t/r^2$.
$\bar{t}_D$ dimensionless time, equal to $t(T_1 + T_2)/r^2(S_{s_1}h_1 + S_{s_2}h_2)$.
t_{in} time at the inflection point, s.
T_1, T_2 transmissivity of the upper and lower layer, respectively; m^2/s.
z vertical coordinate, m.
$Z_D = z/h_1$.
$\hat{z}_D = z/h_2$.
α_1, α_2 hydraulic diffusivity of layer 1 and 2, respectively; m^2/s.
γ_n roots of characteristic equations.
ξ Hankel transform parameter.
$\omega_1 = ((p/\alpha_1) + \xi^2)^{1/2}$.
$\omega_2 = ((p/\alpha_2) + \xi^2)^{1/2}$.

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