Numerical Model for Saturated-Unsaturated Flow in Deformable Porous Media 1. Theory

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A theory is presented for numerically simulating the movement of water in variably saturated deformable porous media. The theoretical model considers a general three-dimensional field of flow in conjunction with a one-dimensional vertical deformation field. The governing equation expresses the conservation of fluid mass in an elemental volume that has a constant volume of solids. Deformation of the porous medium may be nonelastic. Permeability and the compressibility coefficients may be nonlinearly related to effective stress. Relationships between permeability and saturation with pore water pressure in the unsaturated zone may be characterized by hysteresis. The relation between pore pressure change and effective stress change may be a function of saturation. In the transition zone where pore water pressure is less than atmospheric but greater than air entry value, soil moisture diffusivity as used in soil physics and coefficient of consolidation as used in soil mechanics are shown to be conceptually equivalent. It is believed that this model will be of practical interest in studying saturated-unsaturated systems undergoing simultaneous desaturation and deformation.

INTRODUCTION

This work is concerned with the development of a numerical model for simulating groundwater motion in variably saturated deformable heterogeneous porous media. The model considers a general three-dimensional field of fluid flow in conjunction with one-dimensional vertical deformation of the porous medium. It is believed that this model will have general applicability, not only in studying the movement of water in shallow groundwater systems in which the role of the unsaturated zone may be of considerable importance but also in studying a variety of civil engineering and geological engineering problems related to ground settlement.

The foundation for the unified treatment of water flow in variably saturated isothermal porous media was first given by *Buckingham* [1907], who proposed the concept of a capillary potential ψ and showed its functional relation to the moisture content θ in partially saturated soils. *Richards* [1931] combined capillary potential with gravitational potential and showed that Darcy's law, which was originally proposed for saturated porous media, was equally valid in the partially saturated zone. While Buckingham and Richards were mainly concerned with flow in partially saturated soils, *Terzaghi* [1925] was concerned with the engineering properties of soils. He proposed the concept of effective stress in defining the deformation of the soil skeleton. By definition, effective stress is related to capillary potential through the total stress. Thus the concepts of capillary potential, gravitational potential, and effective stress together provide a conceptual basis for developing a mathematical model for the transient motion of groundwater in variably saturated deformable porous media.

Although the theoretical basis has existed for some time, no serious attempt was made to develop a unified treatment for saturated-unsaturated flow in groundwater systems until recently [*Cooley*, 1971; *Freeze*, 1971; *Narasimhan*, 1975; *Neuman*, 1973; *Vauclin et al.*, 1974]. The models of Cooley, Freeze, Neuman, and Vauclin et al. include flow in both the saturated and the unsaturated domains, but these workers do not treat in detail the fundamental stress-strain relationships of the porous medium in response to changes in pore water pressure. Nor do they consider the variation in the permeability of the porous medium in response to changes in effective stress. Other numerical models take into account the stress-strain relationships of the porous medium [e.g., *Sandhu and Wilson*, 1969; *Schiffman and Gibson*, 1964; *Gambolati*, 1973; *Helm*, 1975], but these models are restricted to purely saturated flow. Studies related to the behavior of compacted clays [e.g., *Barden*, 1965; *Bishop and Blight*, 1963; *Bishop and Donald*, 1961; *McMurdie and Day*, 1960] indicate that the relationship between effective stress and pore water pressure in partly saturated fine-grained materials may be quite complex and needs special attention.

The purpose of this work is to develop a numerical model to simulate saturated-unsaturated groundwater flow in which the deformation of the soil skeleton is handled according to *Terzaghi*'s [1925] one-dimensional consolidation theory. The soil deformation may be nonelastic, and the compressibility as well as the permeability characteristics of the saturated soil may be nonlinear functions of effective stress. In addition, the permeability as well as the moisture characteristics of the unsaturated soil may exhibit hysteresis. We will assume that the air phase in the zone of partial saturation is continuous and is everywhere at atmospheric pressure. The numerical model that is developed will be capable of handling a three-dimensional flow region that is composed of heterogeneous isotropic materials and has a complex geometry.

Part 1 of this work discusses the physics of the mathematical model. Part 2 is a detailed account of the numerical algorithm. Part 3 demonstrates the validity of the numerical model by applying it to realistic problems with known experimental or mathematical solutions.

EQUATION OF MASS CONSERVATION

The fundamental equation of transient groundwater motion is an equation of mass conservation. For a flow region which deforms with time the mass conservation equation can be expressed in an integral form as

$$-\int_{\Gamma} \rho_w \mathbf{q}_r \cdot \mathbf{n}\, d\Gamma = \frac{D}{Dt} \int_{V} \rho_w \theta\, dV \tag{1}$$

in which $\mathbf{q}_r$ is the vector flux density of water [*Philip*, 1969] relative to the solid grains and D/Dt denotes the material derivative.

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If the volume element is appropriately small so that ρ_w and θ can be treated as average values over V, then (1) becomes

$$-\int_\Gamma \rho_w \mathbf{q}_r \cdot \mathbf{n}\, d\Gamma = \frac{D}{Dt}(\rho_w \theta V) \tag{2}$$

We now seek to write (2) with ψ as the dependent variable and introduce Darcy's law for the equation of motion in the form [*Philip,* 1969]

$$\mathbf{q}_r = -(k\rho_w g/\mu)\nabla(z + \psi) \tag{3}$$

Note that $z + \psi = \phi$ is the hydraulic head and is related to the *Hubbert* [1940] potential by the relation $\Phi = \phi\rho g$.

If we can assume that $Dz/Dt = 0$, implying that z is fixed during the time interval, and if ρ_w, V, n, and S are functions only of ψ, an assumption which is justified from empirical considerations, then since $\theta = nS$, substitution of (3) into (2) leads to

$$\int_\Gamma \rho_w \frac{k\rho_w g}{\mu}\nabla(z+\psi)\cdot\mathbf{n}\, d\Gamma = \frac{d}{d\psi}(\rho_w VnS)\frac{D\psi}{Dt} \tag{4}$$

Since we are concerned with a deformable porous medium, we have so chosen the volume element V that it contains a constant solid volume V_s and a variable void volume V_v. We will assume that the compressibility of the soil grains can be conveniently neglected in relation to that of the voids and water. A consequence of this choice is that spatial relationships in the surface integral in (4) are all functions of time.

FLUID MASS CAPACITY

We shall now introduce a term for fluid mass capacity M_c defined by

$$M_c = (d/d\psi)(\rho_w VnS) \tag{5}$$

Parameter M_c represents the mass of fluid which the volume element V can absorb due to a unit change in the average value of ψ over V. Using the rule for differentiation of a product we obtain

$$M_c = VnS\frac{d\rho_w}{d\psi} + \rho_w S\frac{d(Vn)}{d\psi} + V\rho_w n\frac{dS}{d\psi} \tag{6}$$

The three terms on the right side of (6) denote three distinct physical phenomena. The first term expresses the ability of water to expand due to changes in hydrostatic pressure, the second represents the deformability of the soil skeleton, and the last represents the desaturation of the pores. We will consider each of these phenomena separately.

Equation of State

The dependence of ρ_w on hydrostatic pressure p is given by

$$\rho_w = \rho_{w0}\exp[\beta(p - p_0)] \tag{7}$$

in which the reference pressure p_0 is usually taken as atmospheric and set to zero. Since water is only slightly compressible, we can let $p = \rho_{w0}g\psi$ without loss of accuracy and obtain

$$\rho_w = \rho_{w0}\exp[\beta\rho_{w0}g\psi] \tag{8}$$

Differentiating (8), we immediately obtain, for the first term on the right-hand side of (6),

$$VnS\, d\rho_w/d\psi = VnS\rho_w\rho_{w0}\beta g = V_s eS\rho_w\rho_{w0}\beta g \tag{9}$$

since $Vn = V_s e$.

Deformation of Soil Skeleton

In the second term of (6) we note that $Vn = V_v$ and $e = V_v/V_s$ and hence

$$d(Vn)/d\psi = V_s\, de/d\psi \tag{10}$$

The dependence of e on ψ is not direct. According to the *Terzaghi* [1925] one-dimensional consolidation theory, e is a function of effective stress σ', and σ' in turn is a function of ψ.

By definition, effective stress is the net stress which acts on the soil skeleton. In one-dimensional consolidation theory, effective stress at a point is defined by the relation [*Lambe and Whitman,* 1969]

$$\sigma' = \sigma - \gamma_w\psi \tag{11}$$

We now make an assumption that is reasonable under most field conditions that the total stress σ at any point in the system does not change with time. Then, the changes in effective stress and pore water pressure are related by

$$\Delta\sigma' = -\gamma_w\Delta\psi \tag{12}$$

Equation (12) suggests that any change in ψ is fully converted to an equivalent change in σ'. Experience in the field of soil mechanics seems to indicate that in the case of fully saturated soils a complete equivalence exists between a change in pore water pressure and a change in mechanical stress. On the other hand, in studying the deformation characteristics of oil reservoir rocks, petroleum engineers often infer that only part of the pore water pressure may be convertible to effective stress [*Robinson and Holland,* 1970]. Dry to extremely dry soils may develop negative pore pressures (moisture suction or moisture tension) in the tens or even hundreds of atmospheres. These capillary stresses are of a thermodynamic nature and have little to do with mechanical stresses.

Between the saturated soils, in which capillary and mechanical stresses may be fully equivalent, and the extremely dry soils, in which capillary and mechanical stresses have no equivalence, lie the partially saturated soils of moderate to high saturation, in which moisture suction is only partly convertible to mechanical stress. To accommodate this situation, a modified form of (11) has been proposed by *Bishop* [1960] and by *McMurdie and Day* [1960]:

$$\sigma' = \sigma - \chi\gamma_w\psi \qquad 0 \le \chi \le 1 \tag{13}$$

While the petroleum engineers refer to χ as boundary porosity, soil engineers sometimes refer to it as Bishop's parameter. Parameter χ has been empirically determined for some compacted soils and has a strong nonlinear relation to saturation. Thus $\chi = \chi(S)$. The functional dependence of χ and S for a compacted soil is given in Figure 1. The relation between σ' and ψ in (13) is schematically represented in Figure 2.

If we assume σ to be constant, (13) yields

$$d\sigma'/d\psi = -\gamma_w\chi' \tag{14}$$

in which $\chi' = (\chi + \psi\, d\chi/d\psi)$. In the light of (14), (10) becomes

$$d(Vn)/d\psi = -V_s\gamma_w\chi'\, de/d\sigma' \tag{15}$$

In soil mechanics literature it is customary to express stress-strain relationships of soils by plotting e versus σ'. A very common boundary condition for field loading is one in which the strains are negligible in the intermediate and minor principal stress directions and all possible strains occur only in the vertical (major principal stress) direction. This type of boundary condition is closely simulated by the uniaxial loading experiments conducted in the laboratory.

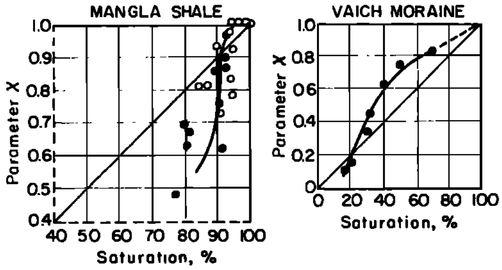

Fig. 1. Relationship between χ and saturation for two compacted soils [after *Bishop and Blight*, 1963].

Figure 3*a* is an example of the relation of e to σ' for a soft clay as determined by uniaxial testing. In this figure, point B represents the state of the soil in situ at the time of sampling. Due to the sampling, transportation, and preparation processes before testing, the soil experiences a reduction in effective stress. Therefore at the commencement of the one-dimensional compression test the state of stress in the soil is represented by point A. As the vertical stress is increased, the sample follows the reloading curve AB. In this region the soil is in a state of 'overconsolidation.' Point B, which represents the maximum effective stress ever experienced by the soil, is the 'preconsolidation' stress of the sample. Once the laboratory loading exceeds the stress level at B, the soil experiences a magnitude of loading never before experienced by it, and the void ratio decreases along the curve BC, which is usually called the 'virgin compression' curve. A soil undergoing such loading is 'normally consolidated.'

At C, further loading is stopped, and a gradual unloading of the soil is commenced. The stress level at C now becomes the new preconsolidation pressure, and the sample, instead of moving along the curve CB, moves along the solid line CD, which is called the 'swelling' or 'rebound' curve. If the sample were to be reloaded at D, it would follow the dotted line connecting D and C, showing a slight hysteresis. For practical purposes, however, this hysteresis can be neglected. The difference between the paths of the virgin and swelling curves shows that the phenomenon of soil deformation is not elastic and that part of the deformation is nonrecoverable. Such nonelastic behavior is exhibited by clays as well as sands. This nonelastic deformation is the prime cause of land subsidence as well as the permanent loss of valuable groundwater storage space in some areas of heavy groundwater withdrawal.

The slope of the curve in Figure 3*a* at any point of interest is called the coefficient of compressibility a_v defined by

$$a_v = -de/d\sigma' \tag{16}$$

in which the negative sign accounts for the fact that e decreases with increasing σ'. Moreover, because of the nonlinear relationship between e and σ', a_v itself is a function of σ'.

Closely related to a_v is the empirical parameter, volumetric compressibility:

$$m_v = -\epsilon_v/\Delta\sigma' \tag{17}$$

where ϵ_v is the volumetric strain given by $\Delta V_v/V_0$. The quantities a_v and m_v are related by

$$a_v = m_v(1 + e_0) \tag{18}$$

Analysis of a large number of uniaxial test data indicates that a plot of e versus log σ' is approximately a straight line (Figure 3*b*). The slope of the best-fitting straight line is called the 'compression index' C_c in the case of the virgin curve and the 'swelling index' C_s in the case of the rebound curve. Parameter C_c usually exceeds C_s by an order of magnitude or more. An advantage of using C_c or C_s to describe stress-strain relationships is that they are dimensionless coefficients, independent of the units of measurement.

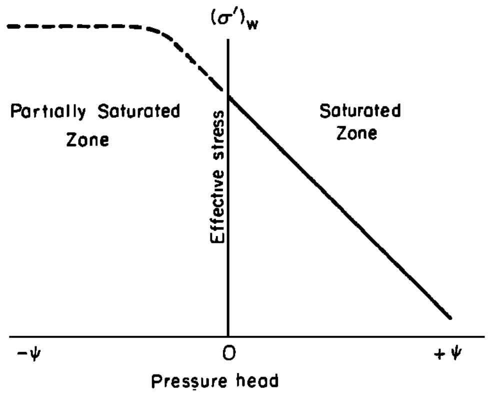

Fig. 2. Schematic representation of effective stress in saturated and unsaturated domains.

Using the chain rule of differentiation, we find that

$$C_c = -\frac{de}{d(\log_{10}\sigma')}$$

$$= -\frac{de}{d\sigma'}\frac{d\sigma'}{d(\ln\sigma')}\frac{d(\ln\sigma')}{d(\log_{10}\sigma')} = 2.303\sigma' a_v \tag{19a}$$

or

$$a_v = C_c/2.303\sigma' \tag{19b}$$

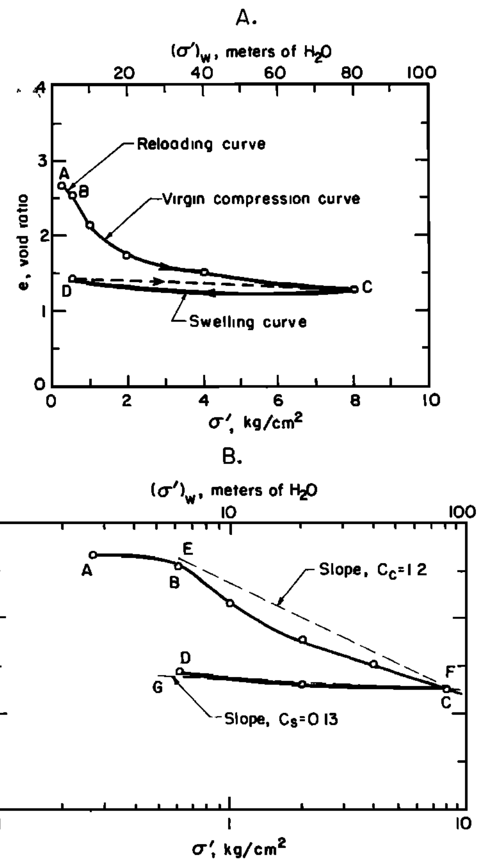

Fig. 3. Variation of void ratio in relation to effective stress. (*a*) Cartesian plot. (*b*) Semi-log plot. (Unpublished data from W. N. Houston, University of California, Berkeley, 1974.)

We are considering e as a function of σ' only. In other words, e changes instantaneously as σ' changes. The experimental data which we use are in fact steady state data in which the soil is allowed to attain equilibrium with each new load before the physical parameters are measured. For many soils the time to attain equilibrium may be relatively small, in which case the assumption of an instantaneous reaction of e to σ' is essentially valid. However, when the soil reacts slowly to changes in loading, accurate simulation would require that e be treated as a function of σ' and t. In the present model, however, we shall ignore the time effects and treat e as a function of σ' only.

We can now evaluate the second term on the right-hand side of (6). Combining (15) and (16), we get

$$\rho_w S\, d(Vn)/d\psi = V_s \rho_w S \gamma_w \chi' a_v \qquad (20a)$$

Or, making use of (19b), we have

$$\rho_w S \frac{d(Vn)}{d\psi} = \frac{V_s \rho_w S \gamma_w \chi' C_c}{2.303\sigma'} \qquad (20b)$$

Desaturation of Pores

The third and last phenomenon that enables a soil to absorb or release water from storage is the change in water saturation, represented by the last term on the right-hand side of (6). Change of water saturation in soils is a thermodynamic process. In extremely dry soils a variation in water saturation may in fact be accompanied by temperature changes. However, in soils of moderate to high water content the temperature does not vary as S changes with ψ. In our model we will neglect temperature effects and assume that S varies only with ψ.

It is well known from laboratory studies that at less than 100% saturation, ψ takes on negative values. In soil physics literature it is customary to refer to such values of ψ as moisture suction or moisture tension. The dependence of S on ψ for $\psi < 0$ is not unique but is characterized by a multiple-valued hysteresis relationship as shown in Figure 4.

If we consider a saturated soil with $\psi = 0$ and apply suction, the soil does not physically desaturate until the applied suction exceeds a critical 'air entry' value ψ_A. The air entry value is a function of the pore diameter of the soil, and for fine-grained sediments and clays it may be of the order of several meters of water or more. In the range $\psi_A < \psi < 0$ the soil remains saturated but has a negative pore pressure. The capillary fringe in natural soils coincides with this range in the values of ψ.

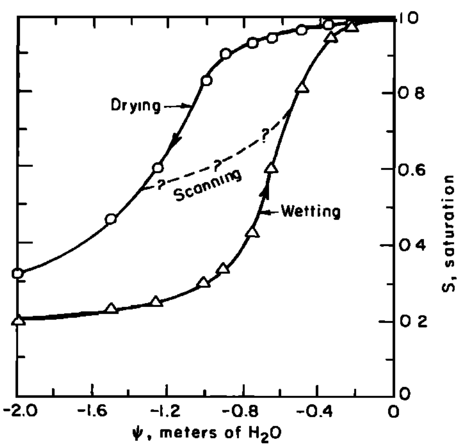

Fig. 4. Variation of saturation with pressure head for Del Monte sand [after *Liakopoulos*, 1965].

Once the threshold air entry value is reached, the S versus ψ relation follows the drying curve. If at any point in the drying curve the process is reversed, a hysteresis effect as shown by the scanning curve in Figure 4 results. The drying and the wetting curves form the boundaries of the hysteresis loop, within which the position of the scanning curve depends on the saturation history.

The slope of the drying, wetting, or scanning curve at any point of interest may be called the 'specific saturation capacity' and is a measure of the ability of the soil to absorb or release water from storage due to saturation changes. If porosity is assumed constant, as is customary in soil physics literature, then

$$n\frac{dS}{d\psi} = \frac{d(nS)}{d\psi} = \frac{d\theta}{d\psi} = C \qquad (21)$$

It is obvious from Figure 4 that $dS/d\psi$ and $d\theta/d\psi$ are strong multiple-valued functions of ψ. Substituting (21) into the last term on the right-hand side of (6) and recognizing that $Vn = V_s e$, we see that

$$V\rho_w n\, dS/d\psi = V_s \rho_w e\, dS/d\psi \qquad (22)$$

Final Expression for M_c

We obtain a final expression for M_c by substituting (9), (20a), and (22) into (6):

$$M_c = V_s \rho_w (S e \rho_{w0} \beta g + S \gamma_w \chi' a_v + e\, dS/d\psi) \qquad (23a)$$

Or using (20b), we can use C_c instead of a_v and write

$$M_c = V_s \rho_w \left(S e \rho_{w0} \beta g + \frac{S \gamma_w \chi' C_c}{2.303\sigma'} + e \frac{dS}{d\psi} \right) \qquad (23b)$$

Note that in (23a) and (23b) the quantities ρ_w, S, and e are all functions of ψ and change continuously with time. The parameter $\chi' = \chi'(S)$ is also a function of ψ, since S is related to ψ.

Meaning of Specific Storage

In hydrogeology literature the coefficient of specific storage S_s is commonly used for saturated soils. Parameter S_s is defined as the volume of water released from a unit bulk volume of the soil per unit change in ψ. Since S_s involves a unit volume element in the saturated zone, we divide (23a) by V, and disregarding the density term and noting that $\chi' = 1$ for full saturation, we obtain in a straightforward manner

$$S_s = \left(n\gamma_w \beta + \frac{\gamma_w a_v}{1 + e} \right) \qquad (24)$$

since $V_s e = V_n$ and $V_s = V/(1 + e)$.

In groundwater hydrology, S_s is invariably treated as constant and independent of ψ, which implies that a_v is effectively a constant. We have already seen from Figures 3a and 3b that in fine-grained sediments, a_v could in fact be a significant function of effective stress. Therefore one should treat a_v (and hence S_s) as a constant only for small changes in the value of effective stress. In young sedimentary basins where land subsidence is known to occur the assumption of constant a_v may not be appropriate. Most land subsidence takes place due to nonrecoverable compaction and loss in storage due to stress changes along the virgin compression curve [*Helm*, 1975]. With increased consolidation, a_v decreases significantly, reflecting not only the decreased rate at which water can be withdrawn from storage but also the permanent loss in groundwater stroage. If we recognize this fact, then the storage space available underground may itself be treated as a valuable resource. It is obvious that for long-range predictions of land subsidence and groundwater management we would have to consider the variation of a_v (and hence S_s) with time. Such a treatment is not possible in the conventional equation used in hydrogeology in which the stress field is completely ignored.

PERMEABILITY

The permeability term in (4) is in general a symmetric second-order tensor. However, in the present work we will restrict ourselves to isotropic materials in which k is a scalar. In the zone of partial saturation, k is directly related to ψ, and this relationship may be characterized by hysteresis, as shown in Figure 5.

In saturated systems the relation between ψ and k is not as direct as in unsaturated systems. In the saturated case, permeability is a function of effective stress which in turn depends on ψ. Experimental studies [*Lambe and Whitman,* 1969] have shown that in fine-grained materials such as clay, k is a pronounced function of e. As is shown in Figure 6, experimental data also show a linear relationship between e and log k. We can therefore represent k as an exponential function of e:

$$k = k_0 \exp\left[\frac{2.303(e - e_0)}{C_k}\right] \qquad (25)$$

where C_k is the best-fitting straight line for the relationship of e versus log $\dot{k}$ (Figure 6). Equation (25) is only one way of representing the dependence of permeability on effective stress. As far as numerical modeling is concerned, one could equally well use any other convenient experimental relationship or simply tabulate k as a function of effective stress.

INITIAL AND BOUNDARY CONDITIONS

The transient movement of groundwater given by (4) is subject to initial and boundary conditions. The initial condition may either be simply hydrostatic or may be represented by a known arbitrary distribution of fluid potential ϕ. The boundary conditions may be prescribed potential, prescribed flux, or of mixed type. In the case of the prescribed potential (Dirichlet) or prescribed flux (Neuman) boundary conditions the potential or flux may be prescribed to vary either as a function of time or as a function of the unknown pressure head ψ.

The phenomenon of a seepage face which is peculiar to saturated-unsaturated flow gives rise to an important mixed boundary condition. On the seepage face the fluid potential is equal to the elevation head and $\psi = 0$. In addition, fluid flux may only leave (but not enter) the porous medium across such a boundary. In a system with an unsaturated zone the seepage face may grow or shrink with time, and hence the actual dimension of the seepage boundary is not known a priori. The seepage face is thus a prescribed potential boundary on which the flux direction is specified.

The phenomena of evaporation and evapotranspiration give rise to another boundary condition peculiar to saturated-unsaturated groundwater systems. A method of handling this condition has been devised by *Neuman et al.* [1975]. The amount of moisture which the atmosphere can take in from the soil is equal to the sum of potential evaporation and potential evapotranspiration and can be determined from micrometeorological data. In addition, there also exist lower

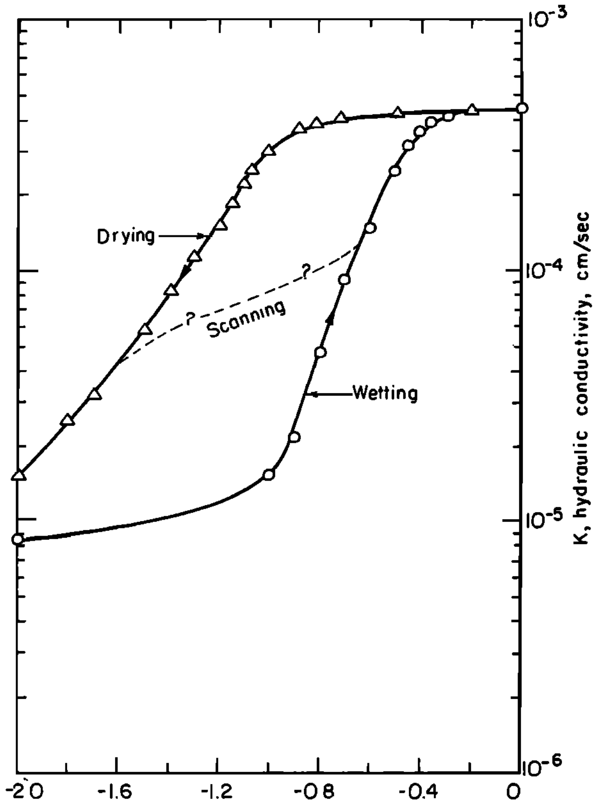

Fig. 5. Variation of hydraulic conductivity with pressure head for Del Monte sand [after *Liakopoulos,* 1965].

limits for the pressure heads that can develop either at the dry soil surface or at plant roots (the wilting pressure of plants). The soil-atmosphere boundary is therefore neither a prescribed potential nor a prescribed flux boundary but is one on which an upper bound for flux and a lower bound for potential are prescribed.

An infiltration boundary constitutes another type of boundary condition similar to evaporation. If the rate of infiltration at the soil surface exceeds the ability of the soil to transmit water, as determined by its saturated vertical permeability, then part of the surface addition must be lost as runoff. Thus an infiltration boundary has an upper limit for the surface flux. In the numerical model developed in the present work, evaporation, evapotranspiration, and infiltration boundaries have not been included. However, there is no conceptual difficulty in incorporating these boundaries in the model.

GOVERNING EQUATION

We can now include a source term G and write the complete governing equation for water movement in a deformable porous medium as

$$\int_\Gamma \rho_w \frac{k\rho_w g}{\mu} \nabla(z + \psi)\cdot \mathbf{n}\, d\Gamma + G = M_c\, D\psi/Dt \qquad (26)$$

in which M_c is given by (23*a*) or (23*b*). Strictly speaking, (26) is a nonlinear equation in which the coefficients ρ_w, k, V, S, n, χ', and e are all functions of the dependent variable ψ. For purposes of numerical solution we could quasi-linearize this nonlinear equation by treating these ψ-dependent coefficients as step functions in time. We also recall that the volume element V which is bounded by the surface Γ has a constant

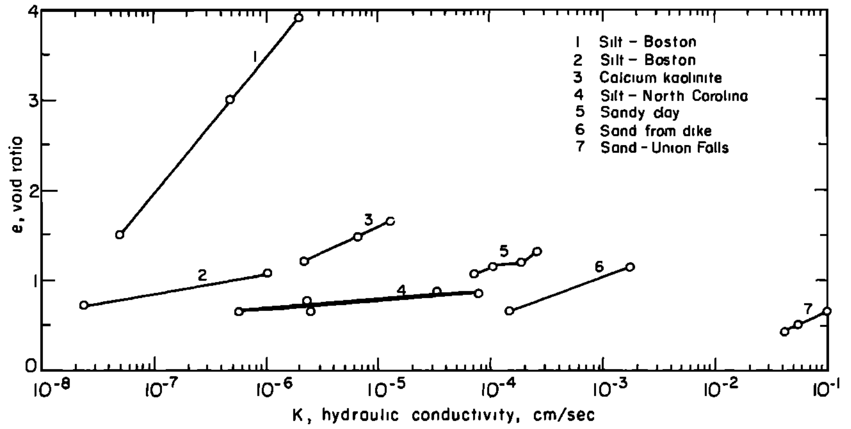

Fig. 6. Variation of hydraulic conductivity with void ratio for some saturated soils [after *Lambe and Whitman*, 1969].

solid volume V_s. Therefore, to be consistent, the spatial relationships in (26) should be treated as step functions in time. It appears from the work thus far that one may neglect this geometric variation without loss of accuracy.

Reduction to a Differential Equation

Equation (25) is an integral form and relates to a finite volume element. If we consider a quasi-linear form of (26) in which V is replaced by $\bar{V}$, where $\bar{V}$ is an appropriate mean volume of the elemental volume over a small interval of time, then we could factor out $\bar{V}$ from both sides of the equation. Further, by letting the elemental volume become arbitrarily small and neglecting the source term we may write

$$\lim_{\bar{V}\to 0} \frac{1}{\bar{V}} \int_\Gamma \rho_w \frac{k\rho_w g}{\mu} \nabla(z + \psi)\cdot \mathbf{n}\, d\Gamma = m_c \frac{\partial \psi}{\partial t} \tag{27}$$

where $m_c = M_c/\bar{V}$ may be called the 'specific fluid mass capacity' of the volume element. Note that the integral on the left-hand side of (27) is the negative of the divergence [*Sokolnikoff and Redheffer*, 1966] of the Darcy velocity defined by (3). Thus (27) reduces in form to the well-known Richards' equation

$$-\text{div}\, \rho_w \mathbf{q} = m_c\, \partial\psi/\partial t \tag{28}$$

where

$$m_c = [\rho_w/(1 + e)](Se\rho_{wo}\beta g + S\gamma_w \chi' a_v + e\, dS/d\psi)$$

θ-Based Equation

A governing equation with ψ as the dependent variable is generally very advantageous in handling heterogeneous flow regions. In such cases the moisture content may vary abruptly in space, but ψ can still be treated as a continuous function. However, a disadvantage of the ψ-based equation is that when k and M_c become strongly dependent on ψ, the equation may become very difficult to solve. This is the case with extremely dry soils. For such problems it is much more convenient to use the volumetric moisture content θ as the dependent variable [see *Braester et al.*, 1972]. In differential form the θ-based equation may be written as

$$\nabla\cdot\rho_w[K\nabla z + D\nabla\theta] = \rho_w\, \partial\theta/\partial t \tag{29}$$

in which $D = K\, d\psi/d\theta$ is the soil moisture diffusivity, which is a function of θ. *Rubin and Steinhardt* [1963] used the θ-based equation to solve infiltration problems in an extremely dry Rehovot sand. The chief disadvantage of the θ-based equation is that it is not suitable for handling heterogeneous media [*Klute*, 1972].

Soil Moisture Diffusivity and Coefficient of Consolidation

In soil physics literature it is customary to treat porosity as a constant in the unsaturated zone. While this assumption may be valid at low saturations, there is a serious difficulty when ψ is in the range $\psi_A \le \psi \le 0$. This range of values of ψ is characteristic of the capillary fringe. In this range the soil remains fully saturated and hence $dS/d\psi = 0$. If now we assume porosity to be constant, then $n\, dS/d\psi = d\theta/d\psi = 0$. Hence D becomes infinite.

In unifying the flows in the unsaturated and saturated zones we can avoid this difficulty by noting that in the capillary fringe, porosity changes with ψ while saturation remains unity. Therefore for $\psi_A < \psi \le 0$ we have $\theta = nS = n$. Hence

$$D = K\frac{d\psi}{d\theta} = K\frac{d\psi}{dn} = \frac{K}{(dn/d\psi)} \tag{30}$$

If the deformation is very small so that the change in void volume is much smaller than the change in bulk volume, then

$$\frac{dn}{d\psi} \approx \frac{\Delta V_v/V_0}{\Delta\psi} \tag{31}$$

Multiplying and dividing by (V_0/V_s) and noting that $(V_0/V_s) = (1 + e_0)$, we obtain from (31)

$$\frac{dn}{d\psi} \approx \frac{\Delta V_v/V_s}{(1 + e_0)\Delta\psi} = \frac{1}{(1 + e_0)}\frac{de}{d\psi} \tag{32}$$

In view of (32) we obtain from (30)

$$D = \frac{K}{(dn/d\psi)} = \frac{K(1 + e_0)}{(de/d\psi)} \tag{33}$$

Also, *Lambe and Whitman* [1969] define the coefficient of consolidation as

$$c_v = \frac{K}{\gamma_w m_v} = \frac{K(1 + e_0)}{\gamma_w a_v} = \frac{K(1 + e_0)}{\gamma_w(-de/d\sigma')} \tag{34}$$

Noting that for a saturated soil, $\Delta\psi = -\Delta\sigma'/\gamma_w$, we have from (34)

$$c_v = \frac{K(1 + e_0)}{(de/d\psi)} \tag{35}$$

Since (35) is identical with (33), we conclude that in the capillary fringe, diffusivity and the coefficient of consolidation are synonymous. Recognition of this equivalence is imperative if flows in the saturated and unsaturated domains are to be unified in a single phenomenological equation.

Limitations of the Mathematical Model

The mathematical model described above is based on a set of assumptions which imposes certain limitations on the model. The first of these assumptions is that the air phase is continuous in the unsaturated zone and remains at atmospheric pressure. If the liquid contains dissolved gas at different pressures, then the present mathematical model becomes inapplicable. Second, the ψ-based equation may become very difficult to solve when the soil is extremely dry, and k as well as S becomes a strong function of ψ. The governing equation (equation (26)) is therefore best-suited to soils of moderate to high saturation.

A third limitation of the present model involves the method of handling soil deformation. The one-dimensional consolidation theory is a simple concept that has been found to be of practical value under many field conditions. However, there may be situations where one will have to consider the complex relation between changes in pore pressure and the general effective stress tensor. A fundamental consequence of this requirement is that we do not know a priori the quantity $de/d\psi$ (which enters into computation of M_c) until we have solved an independent equation relating changes in effective stress to the consequent strains. In order to rigorously solve the problem, we need two equations: one for fluid flow as given by (26) and another for force equilibrium relating changes in effective stress to the deformation (strain) of the soil skeleton. To couple the two equations properly, one would not only need to know the manner in which changes in ψ affect the stress tensor but also the complex three-dimensional stress-strain relationships of variably saturated soils under different boundary conditions. The problem is further complicated by the fact that in a water-saturated soil the deformation is not only governed by changes in stress induced by changes in pore pressure but also by the seepage stresses and the drag forces imposed on individual grains by moving water. The one-dimensional consolidation theory used in this work does not take these complex factors into account.

NOTATION

- a_v coefficient of compressibility (LT^2/M).
- c_v coefficient of consolidation (L^2/T).
- C specific moisture capacity ($1/L$).
- C_c compression index; slope of the e versus log σ' straight line in the normal consolidation region (1).
- C_k slope of the e versus log k straight line (1).
- C_s swelling index; slope of the e versus log σ' straight line in the rebound region (1).
- D soil moisture diffusivity (L^2/T).
- e void ratio (1).
- e_0 void ratio at reference effective stress σ_0' (1).
- g gravitational constant (L/T^2).
- G intensity of source of sink integrated over a finite subregion (M/T).
- k absolute permeability (L^2).
- k_0 absolute permeability at reference void ratio e_0 (L^2).
- K hydraulic conductivity (L/T).
- m_c specific fluid mass capacity (M/L^4).
- M_v coefficient of volumetric compressibility (LT^2/M).
- M_c fluid mass capacity of a finite subregion (M/L).
- n porosity (1).
- $\mathbf{n}$ unit outer normal (1).
- p pressure (M/LT^2).
- p_0 reference pressure (M/LT^2).
- $\mathbf{q}_r$ vector flux density of water relative to solid grains (L/T).
- S saturation (1).
- S_s coefficient of specific storage ($1/L$).
- t time (T).
- V bulk volume of a finite subregion (L^3).
- $\bar{V}$ average bulk volume of a finite subregion during a time interval (L^3).
- V_s volume of solids (L^3).
- V_v volume of voids (L^3).
- z elevation head (L).
- β coefficient of compressibility of water (LT^2/M).
- γ_w specific weight of water (M/L^2T^2).
- Γ surface bounding a finite subregion (L^2).
- ϵ_v volumetric strain (1).
- θ volumetric moisture content (1).
- μ coefficient of viscosity (M/LT).
- ρ_w mass density of water (M/L^3).
- ρ_{w0} mass density of water at atmospheric pressure (M/L^3).
- σ total stress (M/LT^2).
- σ' effective stress (M/LT^2).
- χ Bishop's parameter or boundary porosity, relating effective stress and pore water pressure (1).
- χ' $=(\chi + \psi\, d\chi/d\psi)$ (dimensionless).
- ϕ hydraulic head (L).
- Φ fluid potential or Hubbert potential (M/LT^2).
- ψ pressure head; pore water pressure expressed in equivalent height of water column (L).
- ψ_A pressure head at air entry value (L).

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