Variational Principles for Confined and Unconfined Flow of Ground Water

SHLOMO P. NEUMAN AND PAUL A. WITHERSPOON

University of California, Berkeley, California 94720

Abstract. Finite element techniques are being employed to an increasing degree in solving complicated groundwater flow problems, including problems with a free surface. These methods usually rely on the development of appropriate variational principles. This paper presents several such principles for both confined and unconfined flow.

INTRODUCTION

Finite element techniques for solving problems of groundwater flow are usually based on variational methods. These techniques are finding increasing application because of the ease with which heterogeneous systems that have complicated boundary conditions are handled. They are particularly well suited to the treatment of free surface problems because mesh configurations are easily changed during execution. Further application of the finite element method will often require the development of appropriate variational principles. The purpose of this paper is to present several such principles for both confined and unconfined flow.

The first workers to employ variational methods in the solution of steady state groundwater problems were *Mauersberger* [1965, 1967, 1968*a*, 1968*b*] and *Zienkiewicz et al.* [1966]. Additional steady state solutions have been described by *Taylor and Brown* [1967] and *Finn* [1967] and more recently by *Volker* [1969] and *Neuman and Witherspoon* [1970].

Gurtin [1964*a*, *b*] broadened the application of variational methods by showing how initial conditions may be incorporated in the variational formulation of a problem. As a result, finite element methods have been applied to nonsteady flow of heat [*Wilson and Nickell,* 1966] and ground water [*Javandel and Witherspoon,* 1968, 1969; *Witherspoon et al.,* 1968; *Neuman and Witherspoon,* 1969*a*, *b*].

In all this work, the basic approach is to replace the boundary value problem by an appropriate variational principle whose minimizing function is the solution to the problem. However, it is often possible to solve such problems using direct methods of the calculus of variations without actually developing a variational principle. For example, *Price et al.* [1968] have successfully analyzed one-dimensional dispersion in porous media using the Galerkin method, which does not require the development of a classical variational principle. More recently, *Cavendish et al.* [1969] have shown how to apply the method to initial boundary value problems involving the heat flow equation.

The Galerkin method is applicable to both linear and nonlinear problems, but it has the disadvantage that the boundary conditions cannot be embedded in the variational statement of the problem. A variational principle, on the other hand, presents a more complete representation of the problem in the sense that the boundary (and sometimes, initial) conditions are part of the functional. Thus, whenever possible, it is advantageous to formulate the problem in the form of a variational principle [*Schechter,* p. 236, 1967].

CONFINED FLOW

The simplest case to consider first is the movement of water in a confined flow region, i.e., a completely saturated elastic porous medium that has well defined geometric boundaries. In the following discussion, the flow region is designated by R and its boundary by A. As indicated on Figure 1, A_1 represents those portions of the boundary along which head is prescribed, and A_2 represents the remaining portions across which flux is prescribed. In addition, each point in R whose coordinates are

x_i can be associated with a specific storage $S_s(x_i)$ and a symmetric permeability tensor $K_{ij}(x_i)$. An explanation of the indicial notation used throughout this discussion is given in the appendix.

Let $h(x_i, t)$ be the hydraulic head and $v_i(x_i, t)$ be the Darcy velocity inside the flow region at any given time t. In general, flow in the region R can then be characterized by the equation of continuity

$$\frac{\partial v_i}{\partial x_i} = -S_s \frac{\partial h}{\partial t} \tag{1}$$

and Darcy's law

$$v_i = -K_{ij} \frac{\partial h}{\partial x_j} \tag{2}$$

In addition to the initial condition

$$h(x_i, 0) = h_0(x_i) \tag{3}$$

one will usually encounter boundary conditions involving a prescribed head on A_1,

$$h(x_i, t) = H(x_i, t) \tag{4}$$

and a prescribed flux per unit area on A_2,

$$v_i(x_i, t)n_i(x_i) = V(x_i, t) \tag{5}$$

To obtain a variational principle that corresponds to this problem, it is first necessary to eliminate the initial condition (equation 3) from the above set of equations. This can be accomplished in a manner similar to that of *Gurtin* [1964*b*] by convoluting equation 1 and combining it with the initial condition (equation 3). The result is a single equation

$$1 * \frac{\partial v_i}{\partial x_i} = -S_s(h - h_0) \tag{6}$$

which is completely analogous to equations 1 and 3 and has the advantage that it enables one to incorporate the initial condition directly into a variational formulation of the problem. Here * represents convolution (see appendix).

The above equations have been presented in terms of both head and velocity. However, since hydrologists have traditionally dealt with flow problems in terms of head, we shall first present a variational principle for the particular case where the velocity function has been eliminated from equations 1 through 6. Later, we shall show that there are distinct advantages to an approach that allows one to determine both h and v_i simultaneously.

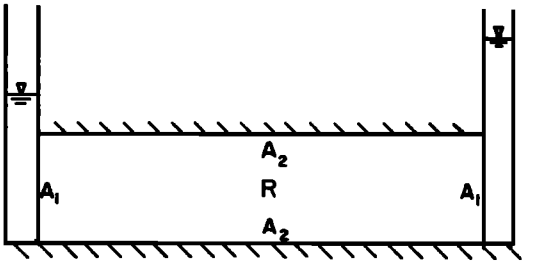

Fig. 1. Diagram of a confined flow region R with prescribed head boundaries A_1 and prescribed flux boundaries A_2.

Variational principle in terms of head. In dealing with problems of flow in porous media, it is customary to define velocity through Darcy's law and thus characterize the problem only in terms of head. If we substitute equation 2 into equations 5 and 6, the problem can be completely defined by the following set of equations:

$$1 * \frac{\partial}{\partial x_i}\left(K_{ij} \frac{\partial h}{\partial x_j}\right) = S_s(h - h_0) \tag{7}$$

$$h = H \quad \text{on} \quad A_1 \tag{4}$$

$$K_{ij} \frac{\partial h}{\partial x_j} n_i = -V \quad \text{on} \quad A_2 \tag{8}$$

This same problem can also be expressed in a different form by the following variational principle:

$$\Omega(h) = \int_R \left[\frac{1}{2} * K_{ij} \frac{\partial h}{\partial x_j} * \frac{\partial h}{\partial x_i} + S_s h * \left(\frac{1}{2} h - h_0\right)\right] dR - \int_{A_1} 1 * (h - H) * K_{ij} \frac{\partial h}{\partial x_j} n_i \, dA + \int_{A_2} 1 * V * h \, dA \tag{9}$$

In equation 9, h now represents all functions $h(x_i, t)$ which, together with their second space derivatives and first time derivatives, are continuous everywhere in R. For this particular problem, such functions are called admissible functions, and one of them will minimize $\Omega(h)$. The concept of the variational principle implies

that this particular minimizing function is the solution to the problem described by equations 7, 4, and 8. In other words, the variation of the functional (9), which is written $\delta\Omega(h)$, vanishes at a particular function $h(x_i, t)$ if, and only if, $h(x_i, t)$ satisfies equations 7, 4, and 8. In effect, we see that the original problem involving a partial differential equation has been transformed into another kind of problem that requires one to search for the minimizing function of equation 9.

The following gives a simple proof for the validity of the above variational principle. A similar procedure can also be used to verify all other variational principles to be given below. Rigorous proofs for these variational principles are given elsewhere (Neuman and Witherspoon, unpublished manuscript, 1970).

Let h^* be some arbitrary function having the same mathematical properties as h, and let λ be a real variable. We replace h in equation 9 by $h + \lambda h^*$ and then determine the first variation of $\Omega(h)$ through

$$\delta\Omega(h) = \frac{d}{d\lambda}\,\Omega(h + \lambda h^*)\bigg|_{\lambda=0} \qquad (10)$$

Using Green's first identity and remembering that the permeability tensor K_{ij} is symmetric everywhere in R, the result becomes

$$\begin{aligned}\delta\Omega(h) = &\int_R \left[-1 * \frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial h}{\partial x_j}\right) + S_s(h - h_0)\right] * h^*\, dR \\ &- \int_{A_1} 1 * [h - H] * K_{ij}\frac{\partial h}{\partial x_j} n_i\, dA \\ &+ \int_{A_2} 1 * \left[K_{ij}\frac{\partial h}{\partial x_j} n_i + V\right] * h^*\, dA\end{aligned} \qquad (11)$$

For the first part of this proof, let us assume that h satisfies equations 7, 4, and 8. In this case, all bracketed terms in equation 11 vanish, which means that $\delta\Omega(h)$ is zero, and therefore h is the desired minimizing function. For the second part of the proof, suppose that h minimizes $\Omega(h)$ such that $\delta\Omega(h) = 0$, which means that the right-hand side of equation 11 vanishes. Since h^* is arbitrary, each of the integrals in equation 11 must vanish independently of the others. Thus, if we assume that the permeability tensor K_{ij} has nonzero values everywhere in R, then each of the bracketed terms in equation 11 must vanish. This means that equations 7, 4, and 8 are satisfied and our proof is complete.

It is of interest to note that in the process of minimizing the functional in equation 9, the functions h that enter into the search are not required to meet any of equations 7, 4, or 8, a priori. All that is necessary is that they satisfy the continuity criteria described earlier. In this sense, $\Omega(h)$ represents the most general variational principle that can be developed for a problem cast only in terms of head.

In some cases, it is desirable to restrict the generality of the variational principle by imposing various constraints on the admissible functions. For example, one may require that in addition to the previously mentioned continuity criteria, the admissible functions also satisfy the boundary condition $h = H$ on A_1. In doing so, the integral over A_1 does not appear as part of the functional; therefore, the expression in equation 9 reduces to a form previously developed by *Gurtin* [1964*b*].

Variational principles in terms of head and velocity. The traditional approach to flow in porous media is to express the governing equations only in terms of head. Velocity does not appear explicitly in these equations, but is determined indirectly through the application of Darcy's law. There are several reasons for this approach. Fundamentally, the concept of hydraulic head leads to mathematical expressions that usually involve only one dependent variable. These expressions have been extensively investigated in several fields of mathematical physics (potential theory, heat flow, diffusion), and the results of these studies have been of great assistance in solving groundwater flow problems. On the other hand, equations that involve both head and velocity (cf. equations 1 through 5) are difficult to solve by the usual methods. Moreover, hydraulic head is easily measured in the laboratory as well as in the field, whereas velocity is not.

It seems to us that if, in addition to head, the hydrologist were also able to solve for v_i at every point of his system, he would have a great deal more useful information. First, the

hydrologist would have a more complete understanding of flow behavior by knowing the magnitude and direction of flow at each point of his system. From such information, he could easily calculate flux across any cross section, including the boundaries. Second, in some problems the development of nonlinear flow attributable to high gradients may result in discrepancies between predicted and observed flow behavior. With velocities at every point, the hydrologist could calculate Reynolds numbers and thus locate those regions where Darcy's law (i.e., equation 2) is not applicable. Third, in dealing with problems of hydrodynamic dispersion, it is well known that the coefficients of dispersion as well as the governing equations are velocity dependent. In this case, a knowledge of velocities is required to solve the problem.

At first glance, it may appear that having a solution in terms of head is always sufficient to enable one to determine velocities using Darcy's law. While this is obviously true for analytical expressions, the same cannot be generally said for numerical solutions where head is obtained at a discrete number of points on a grid. For example, in most finite difference and finite element schemes, it is customary to adopt a linear interpolation of head between neighboring points of the grid. Figure 2*a* illustrates how such an interpolation is applied to a one-dimensional problem. It is apparent that the corresponding gradients are constant between any two nodal points and change abruptly across each point (i.e., the gradient is discontinuous). Consequently, velocities calculated from Darcy's law are also discontinuous as illustrated in Figure 2*b*.

One might attempt to overcome this problem by taking an arithmetic average velocity at each grid point. Mathematical considerations as well as our experience indicate that this procedure is not always satisfactory. A higher order interpolation of head complicates the numerical calculations considerably and still does not always eliminate the problem of discontinuities.

Thus, when one requires satisfactory values for both head and velocity using low order interpolation, the traditional methods are not sufficient. One needs to solve for both h and v_i simultaneously, and this can be accomplished by minimizing the following variational principle:

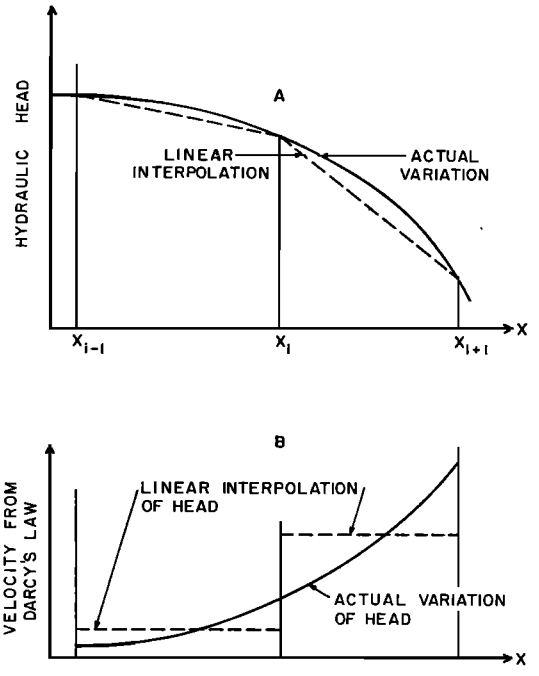

Fig. 2. Velocities determined from Darcy's law and a linear interpolation of head on a one-dimensional grid.

$$\Omega(h, v_i) = \int_R \left[1 * v_i * \frac{\partial h}{\partial x_i} + \frac{1}{2} * v_i * K_{ik}^{-1} v_k - S_s h * \left(\frac{1}{2} h - h_0 \right) \right] dR - \int_{A_1} 1 * (h - H) * v_i n_i \, dA - \int_{A_2} 1 * V * h \, dA \qquad (12)$$

Here K_{ik}^{-1} is the inverse of K_{ji} and is defined by

$$K_{ji} K_{ik}^{-1} = \delta_{jk} \qquad (13)$$

where δ_{jk} is Kronecker delta. In equation 12, h now represents all functions $h(x_i, t)$ which, together with their first space and time derivatives, are continuous everywhere in R, and v_i represents all functions v_i (x_i, t) which are continuous in space and time and have continuous first space derivatives. It can be shown (Neuman and Witherspoon, unpublished manuscript, 1970) that the variation of equation 12 vanishes at a particular set of functions $\{h, v_i\}$ if, and only if, these functions are solutions to

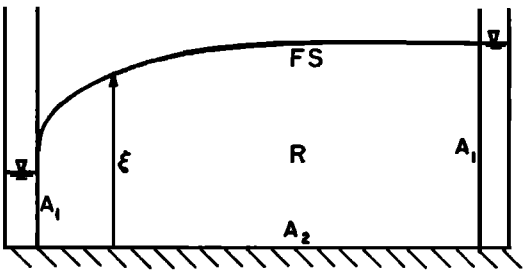

Fig. 3. Diagram of an unconfined flow region R with prescribed head boundaries A_1, prescribed flux boundary A_2, and a free surface FS.

equations 2, 4, 5, and 6. It should be noted that the continuity restrictions on h in equation 12 are less severe than in equation 9.

It may be of interest to note that the convolution approach also enables one to develop a variational principle in terms only of velocity (Neuman and Witherspoon, unpublished manuscript, 1970).

UNCONFINED FLOW

Another case to consider is that of unconfined flow through porous media where it is necessary to treat the free surface as a moving boundary. In this case, the boundary of the flow region consists of three complementary parts: the prescribed head boundary A_1, the prescribed flux boundary A_2, and the free surface FS. The seepage face is included in A_1 (Figure 3).

In dealing with problems involving a free surface, it has been customary to apply the Laplace equation to both steady and nonsteady flow, which means that the effects of compressibility are neglected everywhere below the free surface. However, one can easily find situations in the field where an unconfined aquifer is separated from a confined aquifer by an aquitard. In such cases, the system as a whole is unconfined, yet at least in the aquitard and whatever layers lie below, the effects of compressibility on the nonsteady behavior may be important. Therefore, if we want to treat the entire system as a unit, we must retain S_s in the formulation of the governing equations.

We realize that the effects of capillarity above the free surface can also be significant. This is certainly true in fine grained sediments but becomes much less important in coarse sands and gravels. We shall restrict our analysis to those situations where capillarity can safely be neglected.

Let $\xi(x_1, x_2, t)$ represent the elevation at any time t of a point (x_1, x_2) on FS above the same datum plane from which the hydraulic head is measured. Let $I(x_1, x_2, t)$ be the net rate of infiltration that reaches the free surface from above (considered positive downward).

To avoid complications, we shall describe the behavior of the system only in terms of hydraulic head and ξ. The governing equation then becomes

$$\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial h}{\partial x_j}\right) = S_s\frac{\partial h}{\partial t} \tag{14}$$

In addition to the initial conditions

$$h(x_i, 0) = h_0(x_i) \tag{3}$$

$$\xi(x_1, x_2, 0) = \xi_0(x_1, x_2) \tag{15}$$

one may encounter boundary conditions involving a prescribed head

$$h = H \quad \text{on} \quad A_1 \tag{4}$$

or a prescribed flux per unit area

$$K_{ij}\frac{\partial h}{\partial x_j}n_i = -V \quad \text{on} \quad A_2 \tag{8}$$

Two conditions must be satisfied on the free surface. From our above definition of ξ, one has

$$\xi = h \quad \text{on} \quad FS \tag{16}$$

To obtain the second condition, let us consider the movement of an infinitesimal portion of the free surface dA during an interval of time dt as shown in Figure 4. Let n_i be the unit outer normal. Fluid at the free surface is moving at an average Darcy velocity v_i. The net average rate of infiltration (fluid added to the free surface from above) is I. If Sy is the specific yield of the porous medium, then the volume of moving fluid enclosed between dA at t and at $t + dt$ can be expressed by

$$Sy\ dA\ dL = Sy\frac{\partial \xi}{\partial t}n_3\ dA\ dt \tag{17}$$

The total inflow during dt is

$$(v_i n_i + In_3)\ dA\ dt \tag{18}$$

Equating (17) and (18) and dividing by $dA\ dt$, we arrive at the second boundary condition

$$K_{ij}\frac{\partial h}{\partial x_j}n_i = \left(I - Sy\frac{\partial \xi}{\partial t}\right)n_3 \quad \text{on} \quad FS \tag{19}$$

Since the shape of the flow region changes with time as the free surface moves, the convolution approach is not applicable. This means that initial conditions 3 and 15 cannot be incorporated in the variational principle and must be satisfied independently. As a result, the variational principle will involve both varied and unvaried functions. Such an approach has previously been described by *Rosen* [1954]. Thus, if we vary only h and ξ, while their corresponding time derivatives remain unvaried, the variational principle becomes

$$\Omega(h, \xi) = \int_R \left(\tfrac{1}{2}K_{ij}\frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_i} + S_s h\frac{\partial h}{\partial t}\right) dR$$
$$- \int_{A_1}(h - H)K_{ij}\frac{\partial h}{\partial x_j}n_i\, dA + \int_{A_2} Vh\, dA$$
$$- \int_{FS}(h - \xi)K_{ij}\frac{\partial h}{\partial x_j}n_i\, dA$$
$$- \int_{FS}\xi\left(I - Sy\frac{\partial \xi}{\partial t}\right)n_3\, dA \tag{20}$$

Here h represents all functions $h(x_i, t)$ which, together with their second space and first time derivatives, are continuous everywhere in R, and ξ represents all functions $\xi(x_1, x_2, t)$ which, together with their first time derivatives, are continuous everywhere on FS. It can be shown (Neuman and Witherspoon, unpublished manuscript, 1970) that the variation of equation 20 vanishes at a particular set of functions $\{h, \xi\}$ if, and only if, these functions are a solution to equations 14, 4, 8, 16, and 19.

Although the variational principle given in equation 20 is less general than those presented earlier for confined flow, it is well suited to numerical methods involving the finite element approach. A program based on this functional has already been developed and will be discussed in another paper.

In a manner similar to the treatment of confined flow, a variational principle in terms of h, ξ, and v_i has also been developed and is presented elsewhere (Neuman and Witherspoon, unpublished manuscript, 1970).

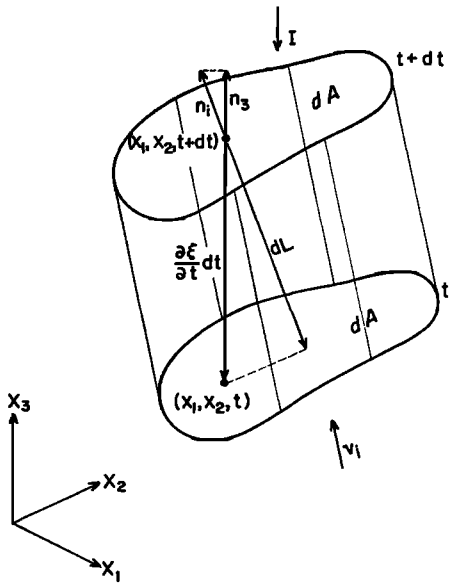

Fig. 4. Infinitesimal element on the moving free surface.

STEADY STATE FLOW

Variational principles for steady state flow are easily obtained from equation 20 simply by dropping the time derivatives. For the case of unconfined flow, one has

$$\Omega(h, \xi) = \int_R \tfrac{1}{2}K_{ij}\frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_i}\, dR$$
$$- \int_{A_1}(h - H)K_{ij}\frac{\partial h}{\partial x_j}n_i\, dA + \int_{A_2} Vh\, dA$$
$$- \int_{FS}(h - \xi)K_{ij}\frac{\partial h}{\partial x_j}n_i\, dA - \int_{FS}\xi I n_3\, dA \tag{21}$$

Except for the last integral, this is identical with a functional previously developed by *Mauersberger* [1965].

For the case of confined flow, the integrals over the free surface do not appear, and the following develops:

$$\Omega(h) = \int_R \tfrac{1}{2}K_{ij}\frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_i}\, dR$$
$$- \int_{A_1}(h - H)K_{ij}\frac{\partial h}{\partial x_j}n_i\, dA + \int_{A_2} Vh\, dA \tag{22}$$

APPENDIX

In the present analysis, we find it convenient to adopt the indicial notation whereby matrices are represented by subscripted quantities which correspond to the individual terms of the matrix. We also use the summation convention according to which repeated indices indicate summation. A more detailed discussion of the indicial notation and the summation convention can be found elsewhere [*Neuman and Witherspoon*, 1969*a*, appendix B].

The three Cartesian coordinates x, y, and z are represented by x_i ($i = 1, 2, 3$, respectively). The components of the outward unit normal to the boundary of R, in the directions of the coordinates x_i, are denoted by n_i.

Let f and g be two functions that are continuous everywhere on R. Then the convolution of f and g is defined as

$$[f * g](x_i, t) = \int_0^t f(x_i, \tau) g(x_i, t - \tau)\, d\tau$$

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