The Distribution of Catchment Coverage by Stationary Rainstorms

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The occurrence of wetted rainstorm area within a catchment is modeled as a Poisson arrival process in which each storm is composed of stationary, nonoverlapping, independent random cell clusters whose centers are Poisson-distributed in space and whose areas are fractals. The two Poisson parameters and hence the first two moments of the wetted fraction are derived in terms of catchment average characteristics of the (observable) station precipitation. The model is used to estimate spatial properties of tropical air mass thunderstorms on six tropical catchments in the Sudan.

INTRODUCTION

The coupled fluxes of heat and moisture at the land surface depend nonlinearly upon the moisture state of the near-surface soil. To define these fluxes accurately over scales of time and space that are large with respect to those of the storm events that recharge the soil moisture demands an event-based moisture accounting whose implementation involves stochastic temporal and spatial definition of the storm.

While much attention has been given to the probability distributions of the characteristics of point rainfall events [see Eagleson, 1978, for example], the spatial uncertainties of rainstorms are still poorly defined, largely because of observational difficulties. For spatial definition of rainstorms, rain gages should have a spacing that is small with respect to the storm dimension. Such dense networks of (recording) rain gages should have a spacing that is small with respect to the storm dimension. Such dense networks of (recording) rain gages are expensive to install and to operate and thus have been used in only a few localities to obtain Eulerian descriptions of small-scale storms [e.g., Huff, 1968; Changnon et al., 1971]. Weather radar provides the Lagrangian view of storms that is essential to the understanding of storm kinematics. Radar studies have concentrated primarily on deterministic taxonomy of storms, however [e.g., Ligda, 1953; Weaver, 1966; Houze, 1969], with somewhat less attention to their statistical description. Notable exceptions are the early work of Kessler and Russo [1963] and of Cole [1964], and the more recent work of Lopez [1977] and Crane [1981]. Satellite imagery affords a detailed instantaneous view of certain characteristics of large-scale precipitation systems but is still in its infancy. Significant contributions have been made in this area by Hobbs [1978], Lovejoy [1982], and others.

An alternative to these methods is to derive the desired spatial storm statistics using a conceptual model of storm occurrences and to estimate the necessary parameters using the plentiful records of point storm rainfall from operational (as opposed to experimental) rain gage networks. The present work follows this approach.

OBSERVED STRUCTURE OF PRECIPITATION SYSTEMS

Storms of diverse meteorological type have been shown by Houze [1969], Austin and Houze [1972], Harrold and Austin [1974], and others to exhibit structural regularities which may be represented by all or a subset of the features shown in Figure 1 [Waymire and Gupta, 1981a]. In their review of observed rainstorm structure, Waymire and Gupta [1981a] define the synoptic areas as those storm areas (cyclonic storms for example) larger than about 10⁴ km². Their duration is from one to several days. The next largest recognizable feature is the large mesoscale area (LMSA) ranging in size from 10³ to 10⁴ km². There may be several of these (i.e., from one to six) within a synoptic area, building and decaying on a time scale of several hours.

Within an LMSA the next largest feature is called the small mesoscale area (SLSA), 10² to 10³ km² in extent, which also builds and decays with a lifespan of a few hours. Within the LMSA, either within or outside an SLSA, is the cell cluster which has the same areal extent of the SLSA and is composed of several convective cells. The convective cells are each from 10 to 30 km² in area, depending upon the storm type.

In this hierarchical structure the rainfall intensity increases as the scale decreases. For example, the average LMSA intensity exceeds that of the background synoptic area in which it is imbedded.

CONCEPTUAL MODEL

Eagleson [1978] has shown the utility, in studies of long-term hydrologic behavior, of a simple conceptual temporal model of point rainfall. His model consists of Poisson arrivals of uniform intensity storms having random and independent total depth and duration. The times between these pulses of uniform intensity contain no precipitation whatsoever. There is need for an equally simple and conceptually compatible model for spatial storm structure. Such a model would enable one to deal with questions of short-term hydrologic behavior in which only a fraction of the catchment is wetted by a single storm.

The spatial equivalent of these temporal pulses of uniform intensity is randomly spaced discs of uniform intensity having random and independent size, total depth, and duration. The space between these discs at any instant contains no precipitation.

Such a model recognizes only one horizontal precipitation scale out of the array presented in Figure 1. We are concerned with the precipitation primarily for purposes of hydrologic modeling where interest is seldom in catchments much larger than 10⁴ km². We may thus confine our modeling to the structure of the LMSA. Consequently, the precipitation disks of our conceptualization will represent either the SLSA's or the cell clusters, depending upon the type of storm being modeled, and the LMSA "background" precipitation between disks will be neglected. This model is sketched in Figure 2,
Fig. 1. Organized structure of precipitation systems [after Waymire and Gupta, 1981a].

where the notation is

- \( A_c \) area of catchment;
- \( A_s \) area of storm (LMSA);
- \( A_{sc} \) area of storm within catchment;
- \( A_o \) area of individual rainy patch (SMSA or cell cluster) within storm.

The objective of this work is to estimate the statistics of the catchment area \( A_{sc} \) that is wetted by a storm, given that rain falls within the catchment. In Figure 2 the instantaneous value of \( A_{sc} \) is shaded.

Due to storm movement and to the life cycle of the cells and cell clusters composing a storm, the surface area being wetted will usually change with time during the storm. This conceptual model will ignore the "smearing" of wetted area produced by storm movement, assuming instead that the rain-producing cell clusters of area \( A_o \) live their lives in a stationary location. New clusters may appear at new, nonoverlapping locations at other times during the storm, however. The purpose of this assumption is to allow use, in parameter estimation, of the body of instantaneous radar and satellite observations of cell clusters. Accordingly, the model is most applicable to the description of nontravelling events, such as tropical air mass thunderstorms. The size of an individual cluster will be assumed constant over its life cycle also. In such stationary cases the point rainstorm depth \( h \) will be identical to the spatially uniform depth \( h_0 \) of the cell cluster.

The key to relating the storm parameters to the observed station rainfall is the total rainstorm volume \( V_{sc} \) within the catchment. This relationship is written

\[
P_s A_c = \bar{V}_{sc} = \sum_{j=1}^{n} V_{s,j}
\]

in which \( P_s \) represents the catchment average total precipitation in a given year and \( V_{s,j} \) is the random number of catchment storms in that year. We use the overbar now to signify areal averages over the catchment. Such averages will be referred to as "catchment" values.

**PARAMETER ESTIMATION**

In practice it is common to have precipitation observations at several widely spaced stations within and near the catchment from which catchment values of precipitation parameters may be estimated using kriging, Thiessen polygon, or other weighting schemes. Through (1), therefore, the "observed" statistics of \( P_s \) will permit estimation of the parameters of the conceptual rainstorm model.

We obtain these statistics by operating on the summation of (1) as outlined by Benjamin and Cornell [1970, pp. 178–179]. If the storm volumes are independent and identically distributed and if \( V_{s,j} \) is independent of the storm volume, then the mean is

\[
E[P_s A_c] = m_{P_s} A_c = m_{A_c} m_{V_{sc}}
\]

and the variance is

\[
\text{Var}[P_s A_c] = \sigma_{P_s}^2 A^2 = m_{A_c} \sigma_{V_{sc}}^2 + \sigma_{V_{sc}}^2 m_{A_c}
\]

Dividing gives

\[
m_{V_{sc}} \left( \frac{\sigma_{V_{sc}}^2}{m_{V_{sc}}} \right) = \left( \frac{\sigma_{V_{sc}}^2}{m_{V_{sc}}} \right) + \left( \frac{\sigma_{V_{sc}}^2}{m_{V_{sc}}} \right)
\]

but

\[
V_{sc} = \sum_{j=1}^{n} V_{s,j} = \sum_{j=1}^{n} h_0 / A_0,
\]

where \( n \) is the number of cell clusters of rainfall volume \( V_0 \) within area \( A_{sc} \). Note that although \( n \) is an integer, there will be clusters that straddle the boundary of the catchment (such as clusters \( A_{01} \) and \( A_{02} \) of Figure 2). To handle these partial areas, we will assume that all clusters whose "centers" lie within \( A_{sc} \) contribute all their precipitation to \( V_{sc} \) and that those whose "centers" lie outside \( A_{sc} \) contribute nothing.

Assuming the cluster volumes to be independent and identically distributed and that \( n \) is independent of \( V_{sc} \), we can operate on (5) to write

\[
E[V_{sc}] = m_0 E[V_0]
\]

\[
\text{Var}[V_{sc}] = m_0 \text{Var}[V_0] + \sigma_0^2 E^2[V_0]
\]

Assuming also that \( h_0 \) and \( A_0 \) are independent random variables, we have

\[
E[V_0] = m_0 m_{A_0}
\]

Fig. 2. Instantaneous view of catchment rainstorm area.
Var \( [V_0] = m_n^2 \sigma_{A_0}^2 + m_{A_0}^2 \sigma_{h_0}^2 + \sigma_{h_0}^2 \sigma_{A_0}^2 \)  

(9)

Now, taking the expected value of (1) and using (6) and (8) gives

\[ A_c E[P_A] = m_m m_m m_{A_0} \]  

(10)

but from Eagleson’s [1978] model of station rainfall

\[ P_{A_i} = \sum_{j=1}^{\nu} h_{ij} \]  

(11)

where \( \nu \) is the number of storms at station \( i \) in a given year. Assuming the station storm depths \( h \) to be independent and identically distributed and that \( \nu \) is independent of \( h \), we can write

\[ E[P_{A_i}] = m_m h_n \]  

(12)

In a homogeneous climate,

\[ E[P_{A_i}] = E[E[P_A]] \]  

(13)

\[ m_n = m_h = m_{h_0} \]  

(14)

Also,

\[ A_{cw} = \sum_{j=1}^{\nu} A_{0j} \]  

(15)

whereupon, with the previous independence assumptions

\[ E[A_{cw}] = m_m A_{0} \]  

(16)

\[ \text{Var} [A_{cw}] = m_m \sigma_{A_0}^2 + \sigma_{h_0}^2 m_{A_0}^2 \]  

(17)

Using (12), (13), (14), and (16), (10) now reduces to the important relationship

\[ E[A_{cw}/A_c] = m_m/m_n \]  

(18)

where \( m_m/m_n \) is always equal to or less than unity.

With (6), (7), (8), (9), (16), and (18), (4) can be written

\[ E[A_c] = m_m \{m_m CV^2[\bar{F}_A] - m_{A_0}^2 \} \]  

\[ \cdot \left( 1 + \frac{\sigma_{h_0}^2}{m_{h_0}^2} \right) - \frac{\sigma_{h_0}^2}{m_n} - \sigma_{h_0}^2 \} \]  

(19)

where

\[ CV^2[\bar{F}_A] = \sigma_{f_A}^2/m_{f_A}^2 \]  

(20)

Similarly, (15) leads to

\[ \text{Var} [A_{cw}/A_c] = \frac{E[A_{cw}/A_c]}{m_m} \left( \text{Var} \left[ A_{cw} / A_c \right] + \frac{\sigma_{h_0}^2}{m_n} E^2 \left[ A_0 / A_c \right] \right) \]  

(21)

Equations (19) and (21) provide the desired mean and variance of the catchment storm-wetted fraction \( A_{cw}/A_c \) in terms of observable characteristics of the station precipitation and statistics of the constituent random variables, \( v_c, A_0, h_0, \) and \( n \). We must now find appropriate distributions for the latter variables.

**DISTRIBUTION OF CATCHMENT STORM ARRIVALS**

The arrival of catchment storms is assumed to be governed by a temporal Poisson process. That is, the probability \( p_{v_c} \) of obtaining exactly \( v_c \) catchment storms during a rainy season of length \( \tau \) is

\[ p_{v_c}(v_c) = \frac{m_{v_c} \exp (-m_{v_c})}{v_c !} \quad v_c = 0, 1, 2, \ldots \]  

(22)

where

\[ m_{v_c} \text{ average number of catchment storms per rainy season,} \]  

\[ \tau \text{ length of rainy season;} \]  

\[ o_v \text{ average arrival rate of catchment storms.} \]

The moments of this distribution give

\[ \sigma_{v_c}^2 = m_{v_c} \]  

(23)

**DISTRIBUTION OF CLUSTER AREAS**

There is conflicting evidence regarding the empirical distribution of rainstorm areas. Houze and Cheng [1977], Lopez [1977], and others give extensive evidence of the lognormal distribution of the horizontal dimension of clouds and radar echoes over a range of 0.1 to 50 km for convective storms from different regions. The two parameters of these distributions are regionally sensitive, however. Crane [1981] uses radar echoes to show that the area of Kansas thunderstorms is exponentially distributed over a range of from 1 to 50 km².

Lovejoy [1982] has recently shown from radar and satellite observations that tropical rain and cloud areas are fractals [Mandlebrot, 1977, 1982] in which the perimeter \( P \) is related to the area \( A \) according to

\[ P \sim A^{D/2} \]  

(24)

where \( D \) is the “fractal dimension.” Fractals are shapes (with \( D > 1 \)) having structure at all scales but having no characteristic length [Mandlebrot, 1977]. Lovejoy [1982] demonstrated this over the range 1 km < \( A_0 < 10^6 \) km² and found \( D = 1.35 \) as is shown in Figure 3.

Kočak [1938] found empirically that the areas of islands obey a hyperbolic probability distribution

\[ \text{Prob} [A > x] = Bx^{-C} \]  

(25)

where \( B \) and \( C \) are positive constants. Mandlebrot [1977, p.
EAGLESON: DISTRIBUTION OF CATCHMENT STORM COVERAGE

The pdf of $A_0$ is now

$$f(A_0) = (D/2)^{D/2} A_0^{-(1+D/2)} \quad \varepsilon \leq A_0 \leq A_{om} \quad (30)$$

and is sketched in Figure 4. It has moments (for $\varepsilon/A_{om} \ll 1$)

$$m_{A_0}/A_{om} = [D/(2-D)][\varepsilon/(A_0)^{D/2}] \quad (31)$$

$$\sigma_{A_0}^2/A_{om}^2 = [D/(4-D)][\varepsilon/(A_0)^{D/2}] - [D/(2-D)]^2(\varepsilon/A_0)^{D} \quad (32)$$

**DISTRIBUTION OF CLUSTER DEPTHS**

Because we have assumed stationary clusters in a climate that is spatially homogeneous, the statistics of cluster depth $h_0$ will be the same as those for station (i.e., "point") rainstorm depth $h$. Eagleson [1978] has shown the gamma distribution $G(\kappa, \lambda)$ to provide a good fit to observed point rainstorm depth. That is

$$f(h) = a(\kappa, \lambda) h^{\kappa-1} e^{-\lambda h} \quad (33)$$

From the moments of this distribution, we can write

$$m_h = m_h = \kappa/\lambda \quad (34)$$

$$\sigma_h^2 = \sigma_h^2 = \kappa/(\lambda)^2 \quad (35)$$

**DISTRIBUTION OF NUMBER OF CLUSTERS IN CATCHMENT STORM**

Cell clusters will be assumed to be Poisson distributed in space [Gupta, 1973]. That is, the conditional probability $P_{NIA_0}(n)$ of obtaining (throughout the course of a storm) exactly $n$ cluster occurrences within the given catchment storm area $A_c$ is

$$P_{NIA_0}(n) = (\nu_c A_c)^n e^{-\nu_c A_c} / [n! (1 - e^{-\nu_c A_c})] \quad (36)$$

where $\nu_c$ is the average number of clusters per unit of catchment storm area (i.e., cluster density).

We note that (36) differs from the usual Poisson distribution in its omission of the possibility that $n = 0$. This is because without a single cell we have no storm, and the present development is conditional upon storm occurrence.

For $\nu_c A_c \gg 1$, the conditional moments of this distribution are

$$E[n|A_c] \approx \nu_c \quad (37)$$

$$\text{Var}[n|A_c] \approx \nu_c \quad (38)$$

The desired marginal moments are given by [Benjamin and Cornell, 1970]

$$E[n] = m_n = \nu_c E[A_c] \quad (39)$$

$$\text{Var}[n] = \sigma_n^2 = \nu_c E[A_c] + \nu_c^2 \text{Var}[A_c] \quad (40)$$

**DISTRIBUTION OF CATCHMENT STORM AREA**

Tropical air mass thunderstorms, which most closely fit the assumptions of this development, are triggered by the diurnal land surface vertical fluxes of heat and water vapor. These stationary instabilities occur seasonally over very large regions of the tropics. That is, $A_c$ may be of order $10^8$ km$^2$ or greater.

For catchment sizes of normal hydrologic interest (say, of order $10^4$ km$^2$ or less) in a homogeneous climate we can expect tropical air mass thunderstorm potential to exist simultaneously over the entire catchment. That is,

$$A_c \equiv A_c \quad (41)$$
whereupon

$$\text{Var} \left[ A_{sc} \right] = 0 \quad (42)$$

For other storm types, of course, this simplification will not apply and $A_{sc}$ is a random variable equal to or less than $A_c$. Calculation of the statistics of $A_{sc}$ in this general case is a tedious geometrical task which will be described fully in a later publication. In brief however, the computation contains the following steps:

To simplify the geometry both the storm and the catchment are represented in the horizontal by circles, as shown in Figure 5. A "storm" is defined as the occurrence of rainfall within $A_c$. The probability that a storm of radius $r_s$ is centered within $r$ of the center of a catchment having radius $r_c$ is

$$p[r|r_c, r_s] = r^2/(r_c + r_s)^2 \quad (43)$$

From geometry we can calculate $A_{sc}$ from the sum or difference of sectors of circles to obtain

$$A_{sc} = g_1(r, r_c, r_s) \quad (44)$$

or

$$r = g_2(A_{sc}, r_c, r_s) \quad (45)$$

The probability that a storm has an area within the catchment equal to or greater than $A_{sc}$, given $r_s$ and $r_c$, is obtained from (43) and (45) as

$$p[A_{sc}|r_s] = g_2^2(A_{sc}, r_c, r_s)/(r_c + r_s)^2 \quad (46)$$

Then we remove the conditional appearance of $r_s$ according to

$$p[A_{sc}|r_c] = \int g_2^2(A_{sc}, r_c, r_s)/(r_c + r_s)^2 f(r_s) \, dr_s \quad (47)$$

where $f(r_s)$ is the pdf of storm radius. The desired mean and variance of $A_{sc}$ can be obtained (for given catchment size) simply from (47).

**DISTRIBUTION OF RAINSTORM AREA**

Lovejoy [1982] demonstrated that up to at least $10^6$ km$^2$ tropical cloud areas have the same fractal dimension as the smaller cell clusters. Assuming that these cloud areas represent the storm scale in which the clusters are imbedded, the pdf of $A_{sc}$ is, by analogy with (30),

$$f(A) = (D/2) \pi^{D/2} A_s^{-(1+D/2)} \quad \varepsilon \leq A_s \leq A_{sc} \quad (48)$$

for $\varepsilon/A_{sc} \ll 1$. From (48) we have, finally,

$$f(r_s) = (D/\pi)^{D/2} r_s^{-(1+D)} \quad (\varepsilon/\pi) \leq r_s \leq (A_{sc}/\pi)^{1/2} \quad (49)$$

**ANALYTICAL SUMMARY**

Using the moments of the above distributions as expressed by (23), (31), (32), (34), (35), (39), and (40), (19) and (21) are reduced to

$$E \left[ \frac{A_{sc}}{A_c} \right] = \left\{ m_{sc} CV^2 [P_A] - \left( 1 + \frac{1}{\kappa} \right) \left( \frac{2-D}{4-D} \right) \frac{A_{sc}}{A_c} \right\} \left\{ 1 + \frac{\sigma_{A_{sc}}^2}{m_{sc}^2} \right\} \quad (50)$$

**LIMITING CASES**

For very large areas of homogeneous rainfall, $A_{sc} \to A_c$, $m_{sc} \to A_c$, $\sigma_{A_{sc}}^2 \to 0$, and $\sigma_{A_{sc}}^2 \to 0$. Using these values in (19) and recognizing that for this one-dimensional case, $P_A \equiv P_A$, we get the relation for station annual precipitation [Eagleson, 1978].

$$\sigma_{p_{sc}}^2/m_{p_{sc}}^2 = (1/m_{sc})[1 + 1/\kappa] \quad (52)$$

For very small storms, $m_{sc} \to 0$, $\kappa \to \infty$, and from (18) and (19)

$$\sigma_{p_{sc}}^2/m_{p_{sc}}^2 \to 0$$

as is expected when each year contains a very large number of independent events.

For tropical air mass thunderstorms, $\sigma_{A_{sc}}^2 = 0$, as discussed above, and (50) and (51) become

$$E \left[ A_{sc}/A_c \right] = m_{sc} CV^2 [P_A] - \left( 1 + 1/\kappa \right) \left( 2-D/(4-D) \right) A_{sc}/A_c \quad (53)$$

$$\text{Var} \left[ A_{sc}/A_c \right] = E \left[ A_{sc}/A_c \right] \left( 2-D/(4-D) \right) \cdot \left[ 1 + D/(4-D) \right] A_{sc}/A_c \quad (54)$$

Referring to (4), we recall that the term $m_{sc}^{-1} E \left[ A_{sc}/A_c \right]$ of (53) represents the contribution to the variance of catchment annual rainfall of variance in the arrivals of catchment storms, while the term $m_{sc}^{-1}(1 + 1/\kappa)(2-D)/(4-D) A_{sc}/A_c$ represents the contribution of variance in storm rainfall volume.

In this case of tropical air mass thunderstorms, as $A_c$ gets large with respect to the scale of the rainfall patches (represented by $A_{sc}$ in (53)), there will be many rainy patches within the catchment during each storm and the variance of storm
rainfall volume will vanish. For \( A_c/A_{\text{ao}} \gg 1 \), therefore, we have the limiting case

\[
E[A_c/A_c] = m_1C_N^2[F(A_c)] \tag{55}
\]

Equation (53) is not applicable for \( A_c/A_{\text{ao}} \ll 1 \) because of our implicit assumption in (5) that the catchment storm is composed of an integer number of clusters.

Equation (53) and (54) are the final relationships of this development. They define the mean and variance of the wetted catchment fraction \( A_{cw}/A_c \) resulting from tropical air mass thunderstorms (or other stationary, synoptic scale events) in terms of two parameters of the constituent rainfall patches, \( A_{\text{ao}} \) and \( D \), and in terms of three parameters of the (spatially homogeneous) station rainfall, \( k, m, \) and \( \sigma_P^2/mA^2 \).

Lovejoy [1982] has found for tropical rainstorms that \( D = 1.35 \) as is shown in Figure 3. Although the radar used in this study had a reliable maximum coverage of about \( 4 \times 10^4 \) \( \text{km}^2 \) the maximum observed rainfall area was about \( 5 \times 10^3 \) \( \text{km}^2 \) (see Figure 3). Accordingly, we will take

\[
A_{\text{ao}} = 5 \times 10^3 \text{ km}^2
\]

The minimum tropical rain area reported by Lovejoy was \( 1 \text{ km}^2 \) which was the resolution of the radar used but these small areas are certainly single convective cells.

The minimum cluster area will be taken as the lower bound of the observed range of SMSAs quoted by Waymire and Gupta [1981a]. That is,

\[
\epsilon = 10^2 \text{ km}^2
\]

Thus \( \epsilon/A_{\text{ao}} \ll 1 \) as was assumed in the analysis.

**Example**

Station Observations

Chan and Eagleson [1980] studied the statistics of annual water yield from six tropical catchments tributary to the Bahr El Ghazal swamps on the Sudanese White Nile (see Figure 6). Using local observations by the Sudanese Meteorological Service, they showed that the distribution (i.e., cdf) of annual precipitation \( P_a \) at the 20 rainfall stations can be closely represented through Poisson arrivals of independent, identically distributed annual rainfall amounts. The probability density function of this distribution is [Eagleson, 1978]

\[
f_{P_a}(y) = \sum_{i=1}^{\infty} \left[ \frac{\lambda y^{y_i-1} e^{-\lambda y_i}}{y_i!} \right] [m_s e^{-m_s y_i}] \quad y > 0
\]

Fig. 6. The Bahr El Ghazal catchment [from Chan and Eagleson, 1980].
where

\[ p[v = 0] = \exp(-m_v) \]

\[ \frac{m_v}{\kappa} = \text{exp} \left(-\frac{m_v}{\kappa}\right) \]

\[ v = \frac{m_v}{\kappa} \]

\[ m_v = 82.9 \]

\[ \kappa = 0.86 \]

\[ m_v = 90.6 \]

\[ \kappa = 0.71 \]

\[ m_v \text{/} \kappa = \text{gamma distribution of station storm depths} \]

\[ \lambda \text{/} \kappa = \text{scale parameter of the gamma distribution of station storm depths, mm}^{-1} \]

A typical example of the quality of this representation is shown by the cumulative distribution functions (cdfs) of Figure 7. In this figure the circles are the observations plotted according to Thomas [1948] and the solid and dashed lines are different estimates of the cumulative of (56):

\[ \text{Prob} \left[ \frac{P_A}{m_{P_A}} < z \right] = e^{-m_v} \]

\[ \cdot \left\{ 1 + \sum_{v=1}^{\infty} \left[ \frac{m_v}{\kappa} \right] P^*[\kappa, m, k] \right\} \]

where \( m_{P_A} \) is the mean annual station precipitation, equal to \( \kappa, m, \lambda \), and \( P^*[\kappa, m, k] \) is Pearson's incomplete gamma function. The mean of this distribution is, of course, unity, while the variance [Eagleson, 1978] is given by (52). The skew is

\[ E[\left( \frac{P_A}{m_{P_A}} - 1 \right)^2] = \left[ 1 + 3/\kappa + 2/\kappa^2 \right] m_v^2 \]

The data available for estimation of the parameters \( \kappa, \lambda \), and \( m_v \) consisted (see Table 1) of (1) relatively long (28 to 73 years) records of annual station rainfall and of the annual number of rainy days and (2) relatively short (3 to 18 years) records of the number of storms with depth \( h \leq 0.1 \text{ mm} \), \( h \leq 1.0 \text{ mm} \), and \( h \leq 10 \text{ mm} \). In addition, descriptions of the rainfall regime report the typical tropical pattern of a single intense late afternoon or early evening shower. For this reason the number of rainy days is used as the estimate of \( m_v \).

For long annual records the normalized variance \( (\sigma_{P_A}/m_{P_A})^2 \) is fairly well determined, and \( \kappa \) may be calculated from (52). For short annual records it may be more accurate to estimate \( \kappa \) from the sample of individual storm depths. Both of these techniques are illustrated in Figure 7, the former yielding the solid line and the latter the dashed line. Due to the long annual records at all stations, \( \kappa \) was estimated exclusively from (52) in this work, and the values are listed in Table 1.

It is of course theoretically possible to extract more information from the station annual rainfall by taking still higher moments of the observations. For example, in the absence of observations of storm characteristics, both \( \kappa \) and \( m_v \) might be estimated from observations of \( P_A \) using (52) and (58). Similarly, we might incorporate a single-parameter spatial correlation structure into our conceptualization of the rainstorm and estimate this parameter through observations of \( P_A \) by taking a third moment of (1). Of course, the practical obstacle

**TABLE 1. Station Rainfall Characteristics**

<table>
<thead>
<tr>
<th>Station Name</th>
<th>Latitude, North</th>
<th>Longitude, East</th>
<th>Altitude, m</th>
<th>Years of Annual Record</th>
<th>From Annuals</th>
<th>Years of Storm Data</th>
<th>From Storm</th>
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<tbody>
<tr>
<td>El Fasher</td>
<td>13°37'</td>
<td>25°20'</td>
<td>730</td>
<td>58</td>
<td>299, 120</td>
<td>34.3</td>
<td>0.22</td>
</tr>
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<td>El Obeid</td>
<td>13°10'</td>
<td>30°14'</td>
<td>570</td>
<td>73</td>
<td>371, 111</td>
<td>34.2</td>
<td>0.49</td>
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<td>En Nahud</td>
<td>12°42'</td>
<td>38°46'</td>
<td>565</td>
<td>64</td>
<td>396, 102</td>
<td>33.8</td>
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<td>Nyala</td>
<td>12°04'</td>
<td>24°33'</td>
<td>675</td>
<td>54</td>
<td>406, 109</td>
<td>42.4</td>
<td>0.88</td>
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<td>Dilling</td>
<td>12°02'</td>
<td>29°28'</td>
<td>670</td>
<td>59, 436</td>
<td>127, 462</td>
<td>46.2</td>
<td>1.19</td>
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<td>Kadugli</td>
<td>11°00'</td>
<td>29°43'</td>
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<td>64, 747</td>
<td>144, 53.2</td>
<td>1.02</td>
<td>8.00</td>
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<td>Talodi</td>
<td>10°37'</td>
<td>30°24'</td>
<td>473</td>
<td>60, 794</td>
<td>150, 39.6</td>
<td>1.00</td>
<td>3.00</td>
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<tr>
<td>Tonga</td>
<td>9°28'</td>
<td>31°03'</td>
<td>390</td>
<td>61, 877</td>
<td>119, 58.8</td>
<td>0.54</td>
<td>15.00</td>
</tr>
<tr>
<td>Fangak</td>
<td>9°04'</td>
<td>30°53'</td>
<td>390</td>
<td>49, 936</td>
<td>119, 59.6</td>
<td>0.21</td>
<td>16.00</td>
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<td>Aweil</td>
<td>8°46'</td>
<td>27°24'</td>
<td>415</td>
<td>41, 901</td>
<td>119, 64.6</td>
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<td>10.00</td>
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<td>Raga</td>
<td>8°28'</td>
<td>25°41'</td>
<td>545</td>
<td>63, 1183</td>
<td>162, 79.6</td>
<td>0.03</td>
<td>10.00</td>
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<td>Meshra el Rek</td>
<td>8°25'</td>
<td>29°16'</td>
<td>427</td>
<td>53, 836</td>
<td>179, 49.5</td>
<td>0.79</td>
<td>11.00</td>
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<tr>
<td>Wau</td>
<td>7°42'</td>
<td>28°01'</td>
<td>435</td>
<td>72, 1126</td>
<td>182, 82.9</td>
<td>0.86</td>
<td>10.00</td>
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<td>Tonj</td>
<td>7°17'</td>
<td>28°45'</td>
<td>430</td>
<td>38, 1056</td>
<td>198, 71.7</td>
<td>0.66</td>
<td>18.00</td>
</tr>
<tr>
<td>Shambe</td>
<td>7°05'</td>
<td>30°46'</td>
<td>405</td>
<td>61, 780</td>
<td>228, 51.1</td>
<td>0.30</td>
<td>6.00</td>
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<tr>
<td>Rumbek</td>
<td>6°48'</td>
<td>29°42'</td>
<td>420</td>
<td>63, 988</td>
<td>217, 63.9</td>
<td>0.48</td>
<td>10.00</td>
</tr>
<tr>
<td>Amadi</td>
<td>5°31'</td>
<td>30°20'</td>
<td>500</td>
<td>39, 1175</td>
<td>185, 80.6</td>
<td>1.00</td>
<td>12.00</td>
</tr>
<tr>
<td>Li Yubo</td>
<td>5°24'</td>
<td>27°15'</td>
<td>600</td>
<td>34, 1498</td>
<td>200, 104.8</td>
<td>1.15</td>
<td>6.00</td>
</tr>
<tr>
<td>Maridi</td>
<td>4°55'</td>
<td>29°28'</td>
<td>750</td>
<td>55, 1385</td>
<td>225, 100.3</td>
<td>0.61</td>
<td>3.00</td>
</tr>
<tr>
<td>Yambio</td>
<td>4°34'</td>
<td>28°24'</td>
<td>650</td>
<td>52, 1429</td>
<td>197, 110.3</td>
<td>0.91</td>
<td>7.00</td>
</tr>
</tbody>
</table>
to both of these extensions is the well-known instability of the estimates of higher moments for all but very large samples (i.e., very long records).

It is interesting to note the uniformity of the station average storm depth $m_h$. This average is plotted in Figure 8 for each of the 20 stations and is remarkably close to a constant 14.3 mm. This is experimental evidence for homogeneity of the convective rainfall mechanism over a large area of the central Sudan. The homogeneity is never perfect, however, so to obtain catchment values of the precipitation parameters, the station values must be averaged areally. It is these areal averages, signified by an overbar in what follows, that must be used in the final equations. That is, $m_i$, $c_i$ and $\kappa$ of the perfectly homogeneous climate of (53) must be replaced by $\bar{m}_h$, $\bar{c}_h$, and $\bar{\kappa}$, respectively.

**Catchment Averages**

The catchment annual precipitation $P_A$ was estimated for each of the common 32 years of station records by Theissen weighting of the station annual totals $P_i$. The Theissen divisions are shown by the straight dashed lines on Figure 6 and indicate that the areal averages are defined for the six catchments in terms of only nine stations. The weights of station $i$ in calculating catchment $j$ averages are given in Table 2. The resulting catchment average values of the important precipitation parameters are summarized in the portion of Table 3 labeled "observed." The last line of the "observed" section of Table 3 contains the average interstation distance for those labeled "observed." The last line of the "observed" section of Table 3 contains the average interstation distance for those labeled "observed.

**Storm Characteristics**

The bottom portion of Table 3 contains the storm characteristics calculated according to the conceptualization present-
TABLE 3. Catchment Precipitation Parameters Using Common 32-Year Station Records

<table>
<thead>
<tr>
<th></th>
<th>Pongo</th>
<th>Naam</th>
<th>Maridi</th>
<th>Tonj</th>
<th>Jur</th>
<th>Loll</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_r$, km$^2$</td>
<td>8,428</td>
<td>11,962</td>
<td>15,390</td>
<td>21,708</td>
<td>54,705</td>
<td>65,338</td>
</tr>
<tr>
<td>$m_r$, days</td>
<td>244</td>
<td>275</td>
<td>244</td>
<td>275</td>
<td>275</td>
<td>214</td>
</tr>
<tr>
<td>$g$</td>
<td>80.3</td>
<td>82.4</td>
<td>72.1</td>
<td>86.4</td>
<td>93.3</td>
<td>74.1</td>
</tr>
<tr>
<td>$m_0$, mm</td>
<td>1.07</td>
<td>0.73</td>
<td>0.54</td>
<td>0.70</td>
<td>1.00</td>
<td>1.76</td>
</tr>
<tr>
<td>$m_h$, mm</td>
<td>14.9</td>
<td>14.6</td>
<td>15.1</td>
<td>14.5</td>
<td>14.9</td>
<td>15.7</td>
</tr>
<tr>
<td>$m_{sh}$, mm</td>
<td>1198</td>
<td>1199</td>
<td>1091</td>
<td>1251</td>
<td>1388</td>
<td>1165</td>
</tr>
<tr>
<td>$\sigma_{P_r}$, mm</td>
<td>99</td>
<td>97</td>
<td>117</td>
<td>127</td>
<td>136</td>
<td>123</td>
</tr>
<tr>
<td>$\Delta x$, km</td>
<td>193</td>
<td>165</td>
<td>133</td>
<td>183</td>
<td>180</td>
<td>185</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Calculated*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{wc}/A_r$</td>
<td>0.59</td>
</tr>
<tr>
<td>$E[ A_{wc}/A_r ]$</td>
<td>0.27</td>
</tr>
<tr>
<td>Var $[ A_{wc}/A_r ]$</td>
<td>0.17</td>
</tr>
<tr>
<td>$m_n$</td>
<td>297</td>
</tr>
<tr>
<td>$m_h$</td>
<td>284</td>
</tr>
<tr>
<td>$\omega_c$, km$^{-2}$</td>
<td>3.6 $\times 10^{-4}$</td>
</tr>
</tbody>
</table>

*With $D = 1.35$, $\varepsilon = 100$ km$^2$, and $A_{wc} = 5 \times 10^3$ km$^2$.

The average number of cell clusters per storm is large so that for large catchments, at least, the central limit theorem can be invoked and the entire distribution of storm area is approximated with these two moments.

Further effort is necessary to verify (53) and (54). To do this, coordinated observations are needed of both storm and station rainfall. In performing this evaluation, it should be remembered that the catchment area, if a natural watershed, is probably also a fractal. In this case, it is very important that $A_r$ and the various $A_0$ be measured with the same resolution.

It should also be possible to extend this work (1) to eliminate some of the restrictive independence assumptions by introducing appropriate correlation structures, and (2) to include other storm types (both stationary and travelling). This will require a more complex analytical structure, however, of the form outlined by Waymire and Gupta [1981b].

**NOTATION**

- $A$: fractal area, km$^2$.
- $A_c$: catchment area, km$^2$.
- $A_{wc}$: wetted area of catchment, km$^2$.
- $A_0$: wetted area of cell cluster, km$^2$.
- $A_{wc}$: maximum cell cluster area, km$^2$.
- $A_r$: total area of storm, km$^2$.
- $A_{wc}$: area of storm within catchment.
- $A_{wc}$: maximum large mesoscale storm area, km$^2$.
- $a$: Theissen weighting coefficient.
- $B$: parameter of distribution.
- $C$: parameter of distribution.
- $D$: fractal dimension.
- $h$: station rainstorm depth, mm.
- $h_0$: point rainfall depth within a cell cluster, mm.
- $i$: counting index.
- $j$: counting index.
- $k$: normalizing parameter, mm$^p$.
- $m( )$: mean of ( ).
- $n$: number of cell clusters in catchment storm.
- $P$: perimeter of cell cluster, km.
- $P_s$: station annual precipitation, mm.
- $p$: value of probability.
- $r$: radial distance from center of catchment.
- $r_s$: radius of storm.
- $\varphi_0$: volume of cluster rainfall, mm km$^2$.
- $\varphi_{wc}$: volume of catchment rainstorm, mm km$^2$.
- $x$: value of fractal area, km$^2$; distance, km.
- $y$: value of annual station precipitation, mm.
- $z$: value of normalized annual station precipitation.
- $\Delta$: increment.
- $\varepsilon$: minimum cell cluster area, km$^2$.
- $\lambda$: parameter of gamma-distributed station storm depth.
- $\nu$: number of station storms per year.
- $\nu_c$: number of catchment storms per year.
- $m$: standard deviation.
- $r$: length of rainy season.
- $\omega_c$: average number of cell clusters per unit area.
- $\omega_a$: average arrival rate of storm catchments.
- $CV[ ]$: coefficient of variation of [ ].
- $E[ ]$: expected value of [ ].
- $f( )$: pdf of ( ).
- $g( )$: function of ( ).
- $G( )$: gamma distribution.
- $P_\alpha[ ]$: Pearson's incomplete gamma function.
- $Var[ ]$: variance of [ ].
- $\Gamma( )$: gamma function.

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**REFERENCES**


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